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ADVECTION & TIME-STEPPING — GEO325M

Previous Lecture:

· Advection eq: $\frac{d\phi}{dt} + v\frac{d\phi}{dx} = 0$

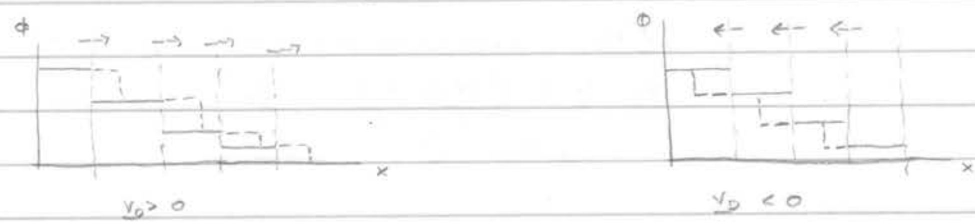
· Method of characteristics - $\phi(x,t) = \phi_0(\underbrace{x-t}_{\text{traveling wave coord}})$

↳ only need inflow BC's

· Steady Discretization

$Pe\,\nabla_D \cdot (\underline{v}_D\,\phi_D) \approx \underbrace{Pe\,\underline{\underline{D}} * \underline{\underline{A}}(\underline{v})}_{\underline{\underline{L}}} * \phi$

$\underline{\underline{L}} * \phi = \ldots$




$a_i = v_i\,\phi_{i-1/2}$ where $\phi_{i-1/2} = \begin{cases} \phi_{i-1}, & v > 0 \\ \phi_i, & v < 0 \end{cases}$ "upwind flux"

Today:

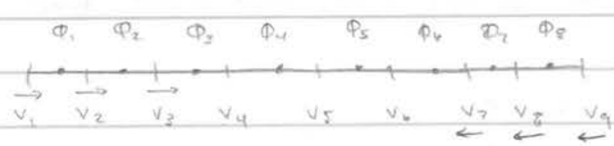
· construct an $\underline{\underline{A}}$ matrix

· transient advection eq $\frac{d\phi_D}{dt_D} + \nabla_D(v_D\,\phi_D) = Da\,\Gamma_D + \phi_D^m(h_D - z_D)$

⇒ theta method

· CFL condition (timestep restriction)

Advection Matrix - Upwind flux




Case 1: $v_i > 0$ $\underline{\underline{A}}^+$

Case I: $v_i > 0$ GT0325M 3/12

$$\underline{\underline{A}}^+ \quad \underline{\underline{A}}^- \qquad \Phi$$

$$\begin{bmatrix} \\ a_2 = v_2 \Phi_1 \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{bmatrix} \overset{v_1}{0} & & & & & & & \\ v_2 & v_2 & & & & & & \\ & v_3 & v_3 & & & & & \\ & & v_4 & v_4 & & & & \\ & & & v_5 & v_5 & & & \\ & & & & v_6 & v_6 & & \\ & & & & & v_7 & v_7 & \\ & & & & & & v_8 & v_8 \\ & & & & & & & v_9^{\,0} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \\ \\ \\ \\ \Phi_8 \end{bmatrix}$$

$a_2 = v_2 \Phi_1$

$a_3 = v_3 \Phi_2$

Case II: $v_i < 0$ $\qquad a_2 = v_2 \Phi_2 \qquad a_1 = v_1 \Phi_1$

$a_3 = v_3 \Phi_3$

To build $\underline{A}(\underline{v})$ we need to select rows of $\underline{A}^+$ and $\underline{A}^-$ according to the entries of $\underline{v}$.

Build positive & negative velocity vectors

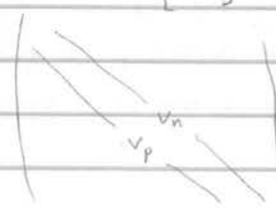
vn = min( v(1:Nx), 0 ); % generating a vector w/all + velocities zero'd out

vp = max( v(2:Nx+1), 0 ); % vector that zeros out all negative vectors

e.g.

$$v = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 7 \\ -8 \\ -11 \end{bmatrix} \qquad vn = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ -8 \\ -11 \end{bmatrix} \qquad vp = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 7 \\ 0 \\ 0 \end{bmatrix}$$

given these vectors

A = 

build w/ sparse diag

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Solution of Transient Advection Eq

PDE: $\frac{\partial \phi_D}{\partial t_D} + Pe \, \nabla_D (v_D \, \phi_D) = 0$

Semi-discrete: $\frac{\partial \phi}{\partial t} + \underbrace{Pe \, \underline{\underline{D}} * \underline{\underline{A}}(\underline{v})}_{\underline{\underline{L}}} + \underline{\phi} = 0$

Theta method:

$$\frac{\phi^{k+1} - \phi^{k}}{\Delta t} + \underline{\underline{L}} \, \underline{\phi}^{?} = 0$$

k is the time level

need to decide the time level of $\underline{\underline{L}} \, \underline{\phi}^{?}$

Theta method: $\underline{\phi} = \theta \, \underline{\phi}^{k} + (1-\theta) \, \underline{\phi}^{k+1}$

substituting:

$$\underline{\underline{I}} \, \underline{\phi}^{k+1} - \underline{\underline{I}} \, \phi^{k} + \Delta t \, \underline{\underline{L}} \left( \theta \, \phi^{k} + (1-\theta) \, \phi^{k+1} \right) = f_s$$

here $\phi^{k}$ is known, $k=0 \Rightarrow$ initial condition; move to rhs

$$\underbrace{\left( \underline{\underline{I}} + \Delta t (1-\theta) \underline{\underline{L}} \right)}_{\underline{\underline{IM}}} \underline{\phi}^{k+1} = \underline{f_s} + \underbrace{\left( \underline{\underline{I}} - \Delta t \, \theta \, \underline{\underline{L}} \right)}_{\underline{\underline{Ex}}} \phi^{k}$$

$$\boxed{\underline{\underline{IM}} \, \phi^{k+1} = f_s + \underline{\underline{Ex}} \, \phi^{k}}$$

generic linear sys for timestepping

$\underline{\underline{A}} \, \underline{x} = \underline{b}$

Properties of Theta Method

For $\theta = 1$: Forward Euler method (explicit)

$\underline{\underline{IM}} = \underline{\underline{I}}$ (diagonal)

$\underline{\underline{Ex}} = \underline{\underline{I}} + \Delta t \, \underline{\underline{L}}$

- b/c $\underline{\underline{IM}}$ is diagonal, no matrix inversion is necessary
- 'explicit update formula': $\underline{\phi}^{k+1} = \underline{f_s} + \underline{\underline{Ex}} * \underline{\phi}^{k}$
- ea timestep is cheap
- conditionally stable: $\Delta t \leq \Delta x / v$ (advection)
- first order $\Delta t \leq \Delta x^2 / 2D_m$ (diffusion)

$D_m$ = diff coeff

$\Rightarrow$ globally limited by fastest flowing region in your domain

$\Rightarrow$ timestep proportional to cell size

$\Rightarrow$ larger timestep $\rightarrow$ larger numerical error

Properties of Theta Method cont'd

3/12/11 For $\theta = 0$: Backward Euler method (implicit)

$\underline{\underline{IM}} = \underline{\underline{I}} + \Delta t\, \underline{\underline{L}}$ (not diagonal!)

$\underline{\underline{Ex}} = \underline{\underline{I}}$

$\Rightarrow$ need to solve a linear sys at ea timestep $\Rightarrow$ implicit

- unconditionally stable (robust)
- first order

For $\theta = \frac{1}{2}$: Crank-Nicholson Method / Trapezoidal rule

$\underline{\underline{IM}} = \underline{\underline{I}} + \frac{\Delta t}{2} \underline{\underline{L}}$ (not diagonal $\rightarrow$ implicit)

$\underline{\underline{Ex}} = \underline{\underline{I}} - \frac{\Delta t}{2} \underline{\underline{L}}$

- second order accurate
- unconditionally stable