# Computing fluxes of gradient fields

In heat flow and many other applications we are concerned with fluxes that are the gradients of scalar potential fields

Fourier's law: $\underline{q} = -\kappa \nabla T$

The discrete approximation of Fourier's law is readily computed using the existing discrete gradient

$$q = -\kappa \, \underline{\underline{G}} * \underline{u} \qquad (\underline{u} = \text{Temperature})$$

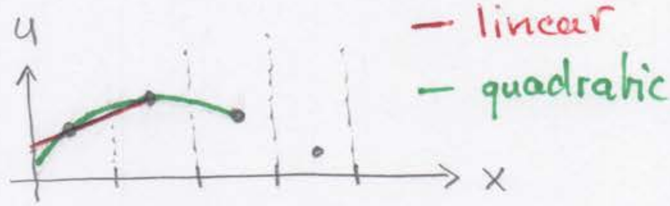
This works well in the interior of the domain, but on the boundary the discrete gradient is zero by construction

$\Rightarrow$ $q = 0$ on boundary

Due to the difficulty of approximating derivative on boundary.

$\Rightarrow$ need to reconstruct the flux across boundary

Option 1




Extrapolate unknown to boundary, equivalent to using a one sided approximation of the derivative

Clearly the flux will depend on the order of the derivative!

We want the one flux that exacty balances the sum of the other fluxes and source/sink terms.

$\Rightarrow$ discrete mass/energy conservation

Option 2: Use the discret balance law in the boundary cell to compute conservative flux.

PDE: $-\nabla\cdot(\kappa\nabla T) = f_s \rightarrow \underline{\underline{L}}\,\underline{u} = \underline{f_s}$

Discrete residual: $\underline{r} = \underline{\underline{L}}\,\underline{u} - \underline{f}_s$

If the discrete equations are satisfied $\underline{r} = \underline{0}$.

In the boundary cells $\underline{r} \neq 0$, because the gradient on the boundary is arbitrarily set to zero!

$\Rightarrow$ non-zero residual contains information about the unknown boundary flux.

## Boundary flux reconstruction

Given the following vectors:

dof-cells = is a vector containing all non-natural boundary cells

dof-faces = is a vector containing the associated bnd faces

Note We assume these vectors are the same length, i.e, each bnd cell has only one face with non-zero flux.

For a problem with both heterogeneous Dirichlet & Neumann BC's the discrete equation is given by

$$\underline{\underline{L}} * \underline{u} = \underline{f_s} + \underline{f}_D + \underline{f}_N$$

so that $\underline{r} = \underline{f}_D + \underline{f}_N$ i.e, $\underline{r} \neq 0$ on those boundary cells

Consider the Neumann bnd ($\underline{f}_D = 0$)

Here we have computed $\underline{f}_n$ by converting the specified bnd flux, qb, into a source term $f_n = q_b A/V$

Now we can reverse this argument and convert the residual $\underline{r} = \underline{f}_n$ back into a flux $q_b = r\, V/A$

On the Neumann bnd this is obvious because $\underline{r} = \underline{f}_n$ but the same is true on the Dirichlet bnd where $\underline{r} = \underline{f}_D$

$\Rightarrow$ Reconstruct the bnd flux at all bnd's as follows

$$\boxed{q(\text{dof-face}) = \underline{\text{sign}}.* \underline{r}(\text{dof\_cells}).*\underline{V}(\text{dof\_cells})./A(\text{dof-face});}$$

where $\text{sign} = \begin{cases} 1, & \text{dof-face} \in [\text{dof-f-xmin}, \text{dof-f-ymin}] \\ -1, & \text{dof-face} \in [\text{dof-f-xmax}; \text{dof-f-ymax}] \end{cases}$

The change in sign on the xmax, ymax bnd's simply indicates that a positive flux on those boundaries is an out flow.