Origin of numerical diffusion - modified equation

Upwind method for advection smoothes the solution → numerical diffusion

How much diffusion is introduced?

How do I need to choose Δt and Δx so that numerical diffusion is less than physical diffusion?

Assume $q > 0 \Rightarrow v = \frac{q}{\phi} > 0$ and consider advective solute transport

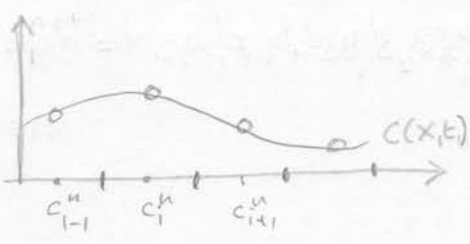
$$\phi \frac{\partial c}{\partial t} + \nabla \cdot (q c) = 0 \quad \rightarrow \quad \frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = 0$$

for the forward Euler ($\theta = 1$) $\quad c_i^{n+1} = c_i^n - \frac{v \Delta t}{\Delta x}(c_i^n - c_{i-1}^n)$

c satisfies the advection eqn to first order

Is there a PDE for example a ADE that is satisfied by c_i to a higher order? If so what is the diffusion term?

Assume $c(x,t)$ is a function that satisfies this unknown PDE and it is identical to c_i^n at grid points $c(x,t) = c_i^n$



Using Taylor series

$$c_{i-1}^n = c(x - \Delta x, t_n) \simeq c - \Delta x\, c_x + \frac{\Delta x^2}{2} c_{xx} - \frac{\Delta x^3}{6} c_{xxx} + \dots$$

$$c_i^{n+1} = c(x_i, t + \Delta t) \simeq c + \Delta t\, c_t + \frac{\Delta t^2}{2} c_{tt} + \frac{\Delta t^3}{6} c_{ttt} + \dots$$

substitute into discrete form

$$\cancel{c} + \Delta t\, c_t + \frac{\Delta t^2}{2} c_{tt} + \frac{\Delta t^3}{6} c_{ttt} = \cancel{c} - \frac{v \Delta t}{\Delta x}\left[\cancel{c} - \left(\cancel{c} - \Delta x\, c_x + \frac{\Delta x^2}{2} c_{xx} - \frac{\Delta x^3}{6} c_{xxx}\right)\right]$$

$$c_t + \frac{\Delta t}{2} c_{tt} + \frac{\Delta t^2}{6} c_{ttt} = v\left[-c_x + \frac{\Delta x}{2} c_{xx} - \frac{\Delta x^2}{6} c_{xxx}\right]$$

$$c_t + v\, c_x = \frac{1}{2}\left(v \Delta x\, c_{xx} - \Delta t\, c_{tt}\right) - \frac{1}{6}\left(v \Delta x^2 c_{xxx} + \Delta t^2 c_{ttt}\right)$$

assuming ratio $\frac{\Delta t}{\Delta x}$ is fixed by $\alpha = v \frac{\Delta t}{\Delta x}$ (CFL number) $\quad \Delta t = \frac{\alpha}{v} \Delta x$

$$c_t + v\, c_x = \frac{1}{2} v \Delta x\left(c_{xx} - \frac{\alpha}{v^2} c_{tt}\right) - v \frac{\Delta x^2}{6}\left(c_{xxx} + \frac{\alpha^2}{v^3} c_{ttt}\right) + \dots$$

If we drop all term $O(\Delta x)$ we recover advection equ (proof that the truncation error is of order Δx)

we only drop term $O(\Delta x^2)$ we obtain

$$c_t + v c_x = \frac{1}{2} v \Delta x \left(c_{xx} - \frac{\alpha}{v^2} c_{tt} \right) + O(\Delta x^2)$$

Need to eliminate c_{tt} term

1) differentiate with respect to t $\quad c_{tt} + v c_{xt} = \frac{1}{2} v \Delta x \left(c_{xxt} - \frac{\alpha}{v^2} c_{ttt} \right)$

2) differentiate with respect to x $\quad c_{tx} + v c_{xx} = \frac{1}{2} v \Delta x \left(c_{xxx} - \frac{\alpha}{v^2} c_{ttx} \right)$

recognizing $c_{xt} = c_{tx}$

$$c_{xt} = - v c_{xx} + \frac{1}{2} v \Delta x \left(c_{xxx} - \frac{\alpha}{v^2} c_{ttx} \right)$$

substitute into ①

$$c_{tt} = - v c_{xt} + \frac{1}{2} v \Delta x \left(c_{xxt} - \frac{\alpha}{v^2} c_{ttt} \right)$$

$$c_{tt} = v^2 c_{xx} + O(\Delta x)$$

so that:

$$c_t + v c_x = \frac{1}{2} v \Delta x \left(c_{xx} - \alpha c_{xx} \right) + O(\Delta x^2)$$

$$= \frac{1}{2} v \Delta x (1 - \alpha) c_{xx} + O(\Delta x^2)$$

Modified equation of upwind method

$$\boxed{\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = \frac{1}{2} v \Delta x \left(1 - v \frac{\Delta t}{\Delta x} \right) \frac{\partial^2 c}{\partial x^2}}$$

Numerical solution solve advection equation to first order but an advection-diffusion equation to second order

$\Rightarrow$ Numerical diffusion $\boxed{D_{Num} = \frac{1}{2} v \Delta x \left(1 - v \frac{\Delta t}{\Delta x} \right)}$

$D_{Num} = 0$ if $CFL = v \frac{\Delta t}{\Delta x} = 1$!