Scaling the Advection Diffusion Eqn

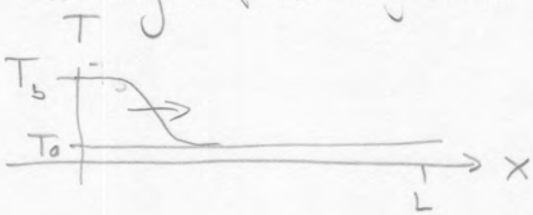
The advection-diffusion eqn for heat transport is

$$\rho c_p \frac{\partial T}{\partial t} + \nabla \cdot \left[\underline{v} \rho c_p T - \kappa \nabla T \right] = \rho H \qquad \text{on } x \in [0, L]$$

For the purpose of the scaling analysis we assume all coefficients are constant and divide by ρc_p to obtain

$$\frac{\partial T}{\partial t} + \nabla \cdot \left[\underline{v} T - k \nabla T \right] = \frac{H}{c_p} \qquad k = \frac{\kappa}{\rho c_p}$$

Consider the simple 1D problem of magma flowing along a channel with velocity v. Consider a hot pulse with $T = T_b$ entering the channel initially at $T = T_0$



Hence we have the IC. $T(x,0) = T_0$

BC: $T(0,t) = T_b$

$\nabla T \cdot \hat{n}\big|_L = 0$ (out flow BC)

How many parameters? $\rho, c_p, |v|, \kappa, \rho, H, T_0, T_b, L$ (9)

Questions: 1) Are all 9 parameters independent

2) Are all terms in PDE equally important?

$\Rightarrow$ scale the variables to make all terms order 1

(need different scales for different problems $\rightarrow$ art)

Scaling has 2 objectives: 1) Reduce # of parameters and identify the governing dimensionless groups.
2) Identify the dominant terms in the PDE.

Scaling and dimensional arguments are at base of most physical reasoning, in particular in non-linear systems.

Scaling analysis

Step 1: Identify characteristic scales for all variables.

dependent variable: T $\quad T_c \in [T_0, T_b]$ $\quad \underline{v}$ $\quad v_c$

independent variables: x $\quad x_c = L$

t $\quad t_c = ?$

Step 2 Define dimensionless variables

$$x_D = \frac{x}{x_c} = \frac{x}{L} \qquad t_D = \frac{t}{t_c} \qquad T_D = \frac{T - T_0}{\underbrace{T_b - T_0}_{\Delta T}} \quad \text{assumes } T_b > T_0$$

Note, it is not a problem that we don't know t_c yet.

Step 3: Substitute into PDE, IC, and BC

First note on derivatives: $\frac{\partial}{\partial t} = \frac{\partial}{\partial (t_c t_D)} = \frac{1}{t_c}\frac{\partial}{\partial t_D}$

PDE $$\frac{\Delta T}{t_c}\frac{\partial T_D}{\partial t_D} + \frac{1}{L}\nabla_D \cdot \left[v\,(T_0 + \Delta T\, T_D) - \frac{k \Delta T}{L}\nabla_D T_D \right] = \frac{H}{c_p}$$

Step 4: Normalize to accumulation term

$$\frac{\partial T_D}{\partial t_D} + \nabla_D \cdot \left[\cancelto{0}{\underbrace{\frac{v_c t_c T_0}{L \Delta T}}} \underline{v}_D + \frac{v_c t_c}{L} v_D T_D - \frac{k\, t_c}{L^2}\nabla_D T_D \right] = \frac{H t_c}{c_p \Delta T}$$

The T_0 term is zero: $\nabla \cdot \left(\underline{v}_D \frac{v_c t_c T_0}{L \Delta T} \right) = \underline{v}_D \cdot \nabla \cancelto{0}{\left(\frac{v_c t_c T_0}{L \Delta T} \right)} + \frac{v_c t_c T_0}{L \Delta T} \cancelto{0}{\nabla \cdot \underline{v}_D}$

constant $\qquad$ incompressible

so that we have

$$\frac{\partial T_D}{\partial t_D} + \nabla_D \cdot \Big[\underbrace{\frac{v_c t_c}{L}}_{\Pi_1} \underline{v}_D T_D - \underbrace{\frac{k t_c}{L^2}}_{\Pi_2} \nabla_D T_D \Big] = \underbrace{\frac{H t_c}{c_p \Delta T}}_{\Pi_3}$$

$\Rightarrow$ Three dimensionless parameter groups

$$\Pi_1 = \frac{L}{T}\frac{T}{L} = 1 \qquad \Pi_2 = \frac{L^2}{T}\frac{T}{L^2} = 1 \qquad \Pi_3 = \frac{\cancel{M}L^2}{T^3}\frac{T^2\Theta}{L^2}\frac{1}{\cancel{\Theta}}\cancel{T} = 1$$

$$H = \frac{W}{kg} = \frac{\cancel{M}L^2}{T^3}\frac{1}{\cancel{M}} \qquad c_p = \frac{J}{kg\,K} = \frac{\cancel{M}L^2}{T^2}\frac{1}{\cancel{M}}\frac{1}{\Theta} = \frac{L^2}{T^2\Theta}$$

Step 5: choose time scale t_c

dimensionless groups suggest 3 time scales:

$\Pi_1 = 1$: $t_c = t_A = \frac{L}{v_c}$ advective time scale

$\Pi_2 \equiv 1$: $t_c \equiv t_D = \frac{L^2}{k}$ diffusive time scale

$\Pi_3 \equiv 1$: $t_c \equiv t_R = \frac{c_p \Delta T}{H}$ reactive time scale

$\Rightarrow$ rewrite PDE in terms of these timescales

$$\frac{\partial T_D}{\partial t_D} + \nabla_D \cdot \Big[\frac{t_c}{t_A} \underline{v}_D T_D - \frac{t_c}{t_D} \nabla_D T_D \Big] = \frac{t_c}{t_R}$$

by choosing t_c to be one of these internal timescales we can reduce the number of dimensionless governing parameters

$\Rightarrow$ choose $t_c = t_D$ "normalizing to diffusion"

$$\frac{\partial T_D}{\partial t_D} + \nabla_D \cdot \Big[\frac{t_D}{t_A} \underline{v}_D T_D - \nabla_D T_D \Big] = \frac{t_D}{t_R}$$

Step 6: Identify standard dimensionless numbers

1) Péclet number: $Pe = \frac{t_D}{t_A} = \frac{v_c L}{k}$

if $Pe \ll 1$ diffusion is dominant transport mechanism

if $Pe \gg 1$ advection is dominant transport mechanism

2) Damköhler number: $Da = \frac{t_D}{t_R} = \frac{H L^2}{c_p \Delta T k} = \frac{\rho H L^2}{k \Delta T}$

if $Da \gg 1$ reaction is faster than diffusion

if $Da \ll 1$ diffusion is faster than reaction

Dimensionless problem:

$$\boxed{\begin{array}{l} \frac{\partial T_D}{\partial t_D} + \nabla \cdot [Pe\, \underline{v}_D T_D - \nabla_D T] = Da \qquad x_D \in [0, 1] \\ \text{IC: } T_D(x_D, 0) = 0 \\ \text{BC: } T_D(0, t_D) = 1, \quad \nabla_D T_D \cdot \hat{x}|_1 = 0 \end{array}}$$