Advection Equation

Porosity evolution equation

$$\frac{\partial \phi_D}{\partial t_D} + \phi_c \nabla_D \cdot [\underline{v}_D \phi_D] = \underbrace{\phi_D^m (h_D - z_D) + Da\, \Gamma_D}_{0} = 0$$

In absence of source terms → advection eqn

If $p_D = h_D - z_D = 0 \Rightarrow \nabla_D \cdot \underline{v}_D = 0$

Rewrite advective flux: $\nabla_D \cdot [\underline{v}_D \phi_D] = \underline{v}_D \cdot \nabla_D \phi_D + \phi_D \cancelto{0}{\nabla_D \cdot \underline{v}_D}$

Advection eqn in standard form

$$\frac{\partial \phi_D}{\partial t_D} + \phi_c \underline{v}_D \nabla_D \phi_D = 0$$

In one dimension

$$\frac{\partial \phi_D}{\partial t_D} + \underbrace{\phi_c v_D}_{v\, =\, \text{const.}} \frac{\partial \phi_D}{\partial x_D} = 0$$

Drop subscripts : $\boxed{\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial x} = 0}$

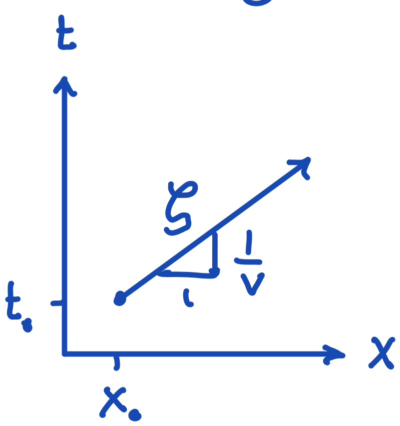
Analytic solution with Method of characteristics

Method of Characteristics

PDE: $\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial x} = 0 \qquad x \in \mathbb{R}$

IC: $\phi(x, t=0) = \phi_0(x)$

Idea: Find a characteristic curve/coordinate, ξ, along which the PDE reduces to an ODE.



$$\phi(x,t) = \phi(x(\xi), t(\xi)) = \Phi(\xi)$$

Total change of Φ along ξ is

$$\frac{d\Phi}{d\xi} = \frac{\partial \phi}{\partial t}\frac{dt}{d\xi} + \frac{\partial \phi}{\partial x}\frac{dx}{d\xi}$$

PDE: $\frac{\partial \phi}{\partial t} \quad + v \frac{\partial \phi}{\partial x} = 0$

By comparison to PDE:

$\frac{d\Phi}{d\xi} = 0 \Rightarrow$ solution does not change along the characteristic curve

$\frac{dt}{d\xi} = 1, \ \frac{dx}{d\xi} = v \Rightarrow \frac{dx}{dt} = v$ "characteristic equ"

solve: $x - x_0 = v(t - t_0)$

from IC: $\phi(x = x_0, t = t_0) = \phi_0(x_0)$

solve char. equ: $x_0 = x - v(t - t_0)$

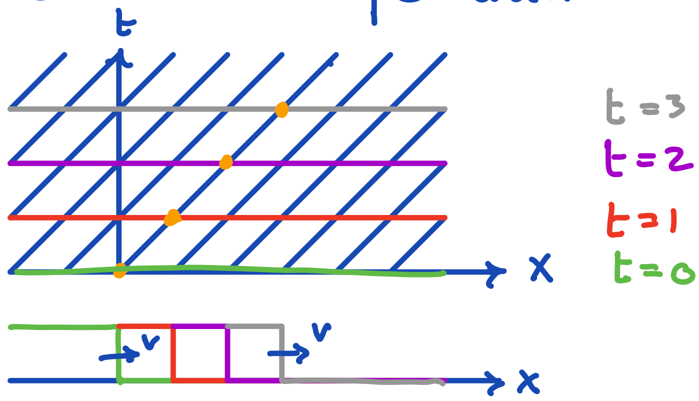
substitute into IC:

$$\boxed{\phi(x,t) = \phi_0(x - v(t-t_0))}$$

gen. solution to adv. eqn. for any $\phi_0(x)$!

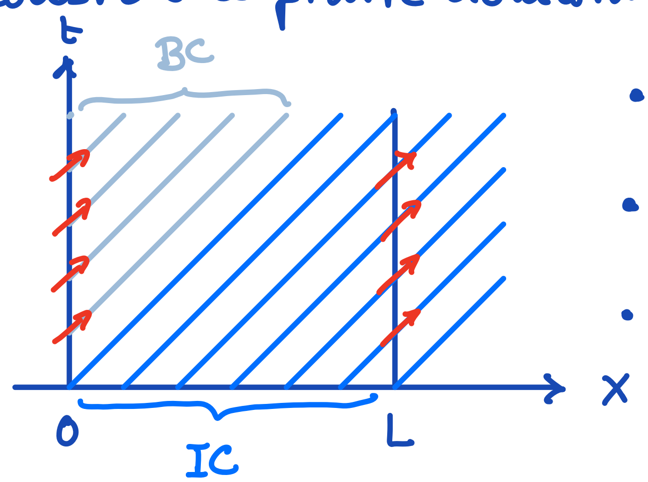
$x - vt$ = travelling wave coordinate ($t_0 = 0$)

The initial profile, ϕ_0, simply translates with constant shape and velocity to the right ($v>0$)



Solution is a front moving with const. velocity.

This was on an infinite domain. Now consider a finite domain

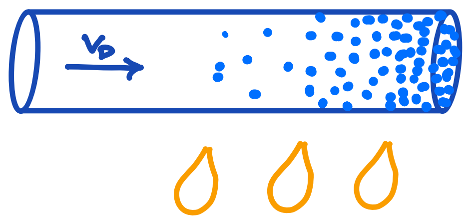


- don't need outflow BC
- need BC on inflow side
- In 2-phase flow the in and outflow bnd depends on phase!

Steady Advection with melting

PDE: $\phi_c \nabla \cdot [\underline{v}_D \phi_D] = \mathcal{D}a\, \Gamma_D$ $x_D \in [0, 1]$

BC: $\phi_D(x_D = 0) = 0$



Pushing a column of ice over a fire and it melts progressively. Here ϕ_c is not clear because the column is initially not molten.

In 1D $v_D = 1$ and $\Gamma_D = 1$ and we choose $\phi_c = \mathcal{D}a$ which corresponds to the final degree of partial melting.

$$\Rightarrow \cancel{\mathcal{D}a}\, \cancel{v_D}^{1} \nabla_D \phi_D = \cancel{\mathcal{D}a}\, \cancel{\Gamma_1}^{1} \Rightarrow \frac{d\phi_D}{dx_D} = 1$$

integrate $\phi_D - \cancel{\phi_0}^{0} = x - \cancel{x_0}^{0}$ $\boxed{\phi_D = x}$

Porosity increases linearly in x-dir.

Discretization of Steady Advection

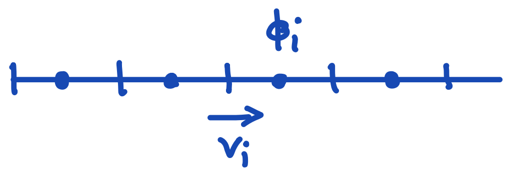
Continuous: $\nabla_D \cdot [\underbrace{\underline{v}_D \phi_D}] = 1$

$$\nabla_D \cdot \underline{q}_D = 1 \qquad \underline{q}_D = \underline{v}_D \phi_D = \text{adv. flux}$$

Discrete: $\underline{\underline{D}}\, \underline{q} = \underline{f}_s$

$$\underline{a} = \text{discrete advective flux vector}$$

How to compute $\underline{a}$ from $\underline{v}$ and $\underline{\phi}$?



$\underline{v} = Nf$ by 1 on faces

$\underline{\phi} = N$ by 1 in cells

Advection matrix: $\underset{Nf\cdot 1}{\underline{a}} = \underset{Nf\cdot N}{\underline{\underline{A}}}\ \underset{N\cdot 1}{\underline{\phi}}$

$\underline{\underline{A}}$ is a Nf by N matrix that computes $\underline{a}$ from $\underline{\phi}$. Shape is same as $\underline{\underline{G}}$

$\underline{\underline{A}} = \underline{\underline{A}}(\underline{v})$ must be a function of $\underline{v}$!

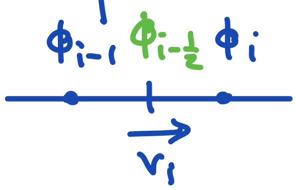
In that case we discretize:

$$\nabla_D \cdot [\underline{v}_D \phi_D] = 1$$

$$\underline{\underline{D}}\, \underline{\underline{A}}(\underline{v})\, \underline{\phi} = \underline{f}_s \qquad \underline{\underline{L}}\, \underline{\phi} = \underline{f}_s \qquad \underline{\underline{L}} = \underline{\underline{D}}\, \underline{\underline{A}}(\underline{v})$$

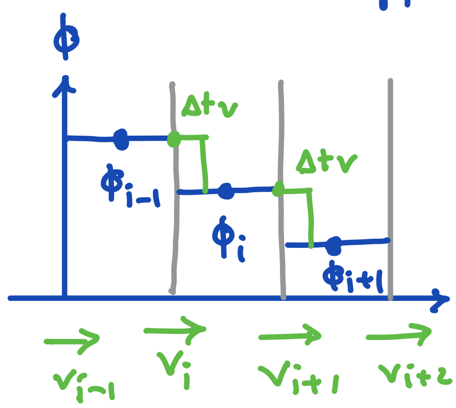
Construction of $\underline{\underline{A}}$

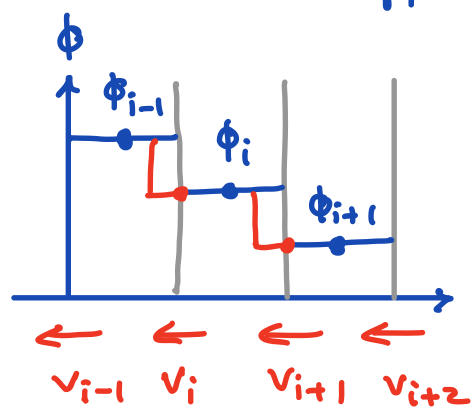
The purpose of $\underline{\underline{A}}$ is to estimate ϕ on the cell faces and to multiply by $\underline{v}$



$$a_i = v_i \, \phi_{i-\frac{1}{2}}$$

How do we approximate $\phi_{i-1/2}$? pp





all $v_i > 0$

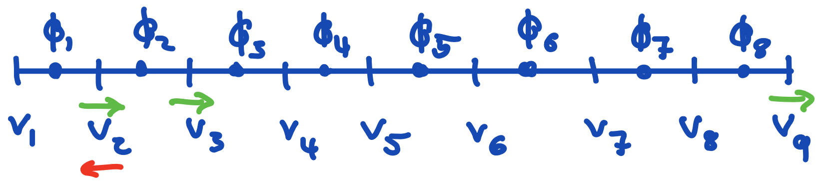
$v_i > 0$: $\phi_{i-1/2} = \phi_{i-1}$

$v_i < 0$: $\phi_{i-1/2} = \phi_i$

From the analytic solution we know $\phi_{i-1/2}$ depends only on upstream/upwind ϕ values

Natural choice:
$$\boxed{\phi_{i-\frac{1}{2}} = \begin{cases} \phi_{i-1} & v > 0 \\ \phi_i & v < 0 \end{cases}}$$

Construction of $\underline{\underline{A}}$



$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} = \underbrace{\begin{bmatrix} v_1 & & & & & & & \\ v_2 & v_2 & & & & & & \\ & v_3 & v_3 & & & & & \\ & & v_4 & v_4 & & & & \\ & & & v_5 & v_5 & & & \\ & & & & v_6 & v_6 & & \\ & & & & & v_7 & v_7 & \\ & & & & & & v_8 & v_8 \\ & & & & & & & v_9 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \\ \\ \\ \\ \phi_8 \end{bmatrix}$$

$a_2 = v_2 \phi_1$
$a_3 = v_3 \phi_2$

$a_1 = v_1 \phi_1$

$$\underline{\underline{A}} = \underline{\underline{A}}^{+} + \underline{\underline{A}}^{-}$$

To build $\underline{\underline{A}}$ we need to select appropriate rows of $\underline{\underline{A}}^{+}$ and $\underline{\underline{A}}^{-}$ according to sign of the corresponding entry of $\underline{v}$.

Bulid pos. and neg. velocity vectors:

$\underline{vn} = \min(\underline{v}(1:Nx), 0)$

$\underline{vp} = \max(\underline{v}(2:Nx+1), 0)$

$$\underline{v} = \begin{bmatrix} 4 \\ -7 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix} \qquad \underline{vn} = \begin{bmatrix} 0 \\ -7 \\ 0 \\ -1 \\ -2 \\ 0 \end{bmatrix} \qquad \underline{vp} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

given these two vectors we build $\underline{\underline{A}}(\underline{v})$ as

$$\underline{\underline{A}} = \begin{array}{cc} & 0 \\ -1 & \begin{bmatrix} \ddots & & \\ \ddots & vn & \\ & vp & \ddots \end{bmatrix} \end{array}$$

assemble with spdiags