

Compute Fluxes of Gradient Fields

We regularly need to compute fluxes of the gradients of scalar potential fields.

$$\mathbf{q} = -\kappa \nabla h$$

h = scalar potential

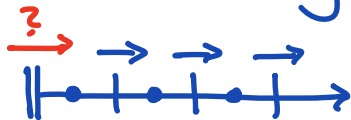
$$\Rightarrow \nabla \times \mathbf{q} = 0$$

Discrete approximation:

$$\mathbf{q} = -\kappa \underline{\underline{\underline{G}_h h}}$$

This works in the interior of the domain, but

on boundary $\underline{\underline{\underline{G}_h h}}$ is zero by construction.



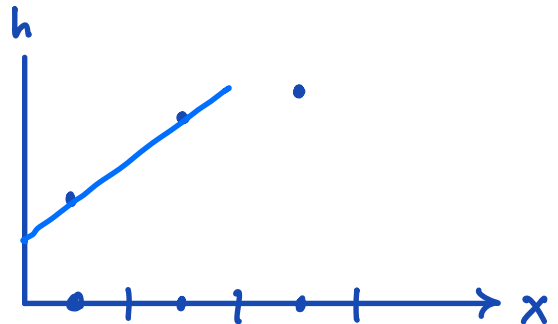
\Rightarrow need to reconstruct boundary flux

Option 1: Extrapolate to bnd

Equivalent to using a

one-sided derivative

Problem: lose discrete



conservation because error in interpolation

Option 2: Reconstruct from discrete balance

Idea: Use the discrete balance in the bnd cell to compute the exact bnd flux required for conservation.

Consider a discrete linear system

$$\underline{L} \underline{u} = \underline{f}_s \quad \underline{u} = \text{unknown}$$

Discrete residual of equation

$$\underline{r}(\underline{u}) = \underline{L} \underline{u} - \underline{f}_s$$

If the discrete equations are satisfied $\underline{r} = \underline{0}$.

In the bnd cells $\underline{r} \neq 0$ because \underline{G} arbitrarily sets the gradient/flux to zero.

\Rightarrow non-zero residual in the bnd cells

contains information about the boundary flux!

Consider a system of flux boundary

$$\underline{L} \underline{u} = \underline{f}_s + \underline{f}_n$$

with residual: $\underline{r} = \underline{L} \underline{u} - \underline{f}_s = \underline{f}_n$

$$\Rightarrow \underline{r} = \underline{f}_n$$

The residual on bnd is equal to the r.h.s. vectors due to the boundary fluxes!

Entries of \underline{f}_n on bnd are: $f_n = q_b \frac{A}{V}$

If we are given $\underline{r} = \underline{f}_n$ we can reverse this

argument and solve for flux: $q_b = f_n \frac{V}{A} = r \frac{V}{A}$

This also works on Dirichlet bnd so that the boundary flux is generally given by:

$$|q_b| = |r| \frac{V}{A}$$

note: up to a sign!

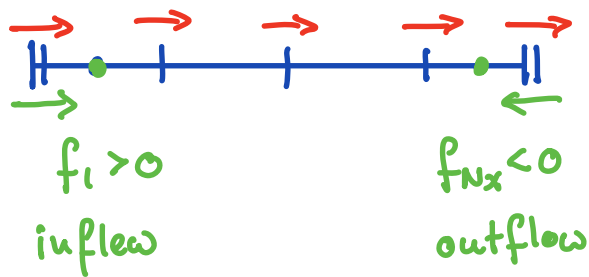
Note: We assume only 1 face of bnd cell has flux!

Sign change:

We want q_b to have sign that fits with

the rest of fluxes computed as $q = -K \nabla h$

These q 's are positive if they point in x -dir.



Need to change sign
on x_{\max} bnd.

Implementation

In function `comp_flux_res.m` we will compute boundary fluxes as follows:

Define two vectors:

dof-cell : column vector containing
all bnd cells

dof-face : column vector containing
all associated bnd faces

These vectors are same length because we assume only one face is associated with each bnd cell.

Compute all bnd fluxes together in one line

$$q(\underline{\text{dof-face}}) = \underline{\text{sign}} .* \underline{r}(\underline{\text{dof-cell}}, \underline{u}) .* \underline{V}(\underline{\text{dof-cells}}) ./ \underline{A}(\underline{\text{dof-face}})$$

where $\underline{\text{sign}} = \begin{cases} 1, & \text{dof-face} \in \text{max-bnd} \\ -1, & \text{dof-face} \in \text{min-bnd} \end{cases}$

You can use `ismember.m` to detect bnd.