Introduction to numerical methods

We would live to solve incompressible flow

$$-\nabla \cdot (K \nabla h) = f$$

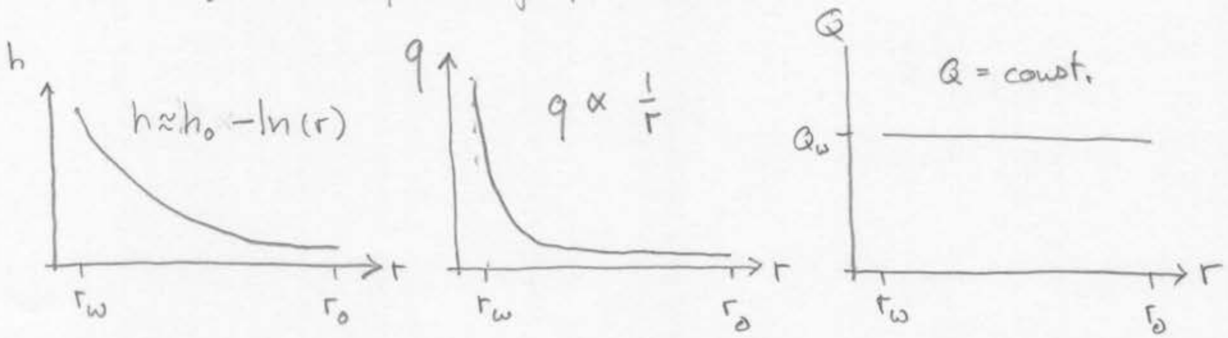
subject to suitable BC's Consides const rate inj. well

PDE: $$-\frac{d}{dr}\left(r \frac{dh}{dr}\right) = 0 \quad \text{on} \quad r \in [r_w, r_o]$$

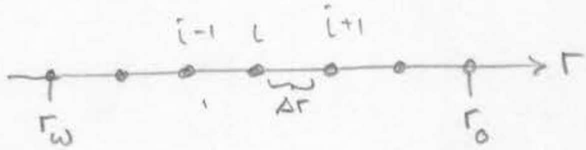
BC: $$Q_w = A_w q(r_w) = -A_w K \left.\frac{dh}{dr}\right|_{r_w} \Rightarrow \left.\frac{dh}{dr}\right|_{r_w} = \frac{-Q_w}{A_w K} \qquad A_w = 2\pi r_w H$$

"Injection with const rate into a well with radius r_w."

$h(r_o) = h_o$ "fixed farfield head"



Finite Difference Discretization



Standard 2^{nd}-order FD ap

$$\left.\frac{dh}{dx}\right|_i \approx \frac{h_{i+1} - h_{i-1}}{2\Delta r} \qquad \left.\frac{d^2h}{dx^2}\right|_i \approx \frac{h_{i+1} - 2h_i + h_{i-1}}{\Delta r^2}$$

on left boundary: $$\left.\frac{dh}{dx}\right|_1 = \frac{-3h_1 + 4h_2 - h_3}{2\Delta r}$$

To use FD approximation we need to rewrite equ

$$-\frac{d}{dr}\left(r \frac{dh}{dr}\right) = -r \frac{d^2h}{dr} - \frac{dh}{dr} = 0$$

demo_compare_FD_FV.m

Observations: 1) Surprisingly large errors in head because we need to resolve derivative in boundary layer

2) Q is not constant, because scheme is not conservative
$\Rightarrow$ transport of solute with wrong speed

Finite differencing conservation form

Coupling between flow & transport ⇒ conservative discretization

Key is to discretize the PDE in conservation form, i.e. with divergence intact.
$$-\nabla\cdot(K\nabla h) = f$$

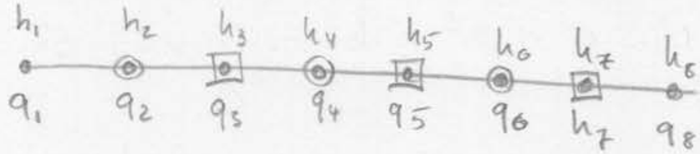
or in one dimension
$$-\frac{d}{dx}\left(K\frac{dh}{dx}\right) = f$$

Rewrite as "div-grad" system of two first order equations by introducing volumetric flux q:

1) $\nabla\cdot\bar{q} = f \xrightarrow{1D} \frac{dq}{dx} = f \xrightarrow{FD} \frac{q_{i+1}-q_{i-1}}{2\Delta x} = f_i$

2) $\bar{q} = -K\nabla h \xrightarrow{1D} q = -K\frac{dh}{dx} \xrightarrow{FD} q_i = -K_i\frac{h_{i+1}-h_{i-1}}{2\Delta x}$

Here the two unknowns h_i and q_i are co-located at x_i.



substitute q_i into approx for ①

$$\frac{1}{2\Delta x}\left[-K_{i+1}\frac{h_{i+2}-h_i}{2\Delta x} + K_{i-1}\frac{h_i-h_{i-2}}{2\Delta x}\right] = f_i$$

$$\frac{1}{4\Delta x^2}\left[-K_{i+1}h_{i+2} + (K_{i-1}-K_{i+1})h_i - K_{-1}h_{i-2}\right] = f_i$$

equation for $i=4$ involves h_2, h_4, h_6 ○

equation for $i=5$ involves h_3, h_5, h_7 □

we have a wide stencil & even and odd dof's don't communicate!

f we reorder $\underline{h}$ so that all odd dof's come first $\underline{\underline{L}}$ becomes block diagonal.

$$\underline{h} = \begin{bmatrix} h_1 \\ h_3 \\ h_5 \\ \vdots \\ h_2 \\ h_4 \\ h_6 \end{bmatrix} \qquad \underline{\underline{L}} = \begin{bmatrix} \boxed{\underline{\underline{L}}_{odd}} & \times \\ \times & \boxed{\underline{\underline{L}}_{even}} \end{bmatrix}$$

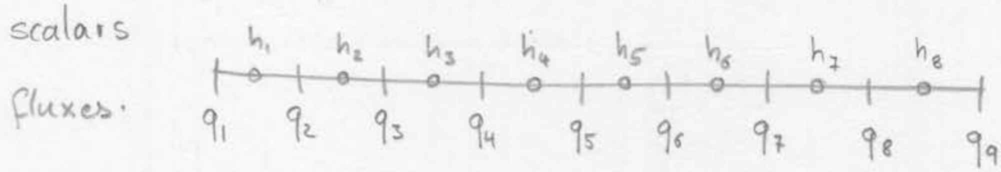
⇒ checkerboard oscillations

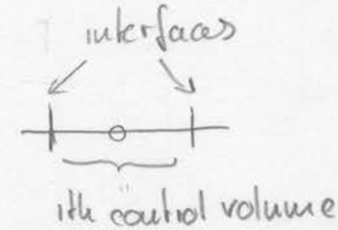
Note only coupling is through BC's!

$\boxed{\text{demo\_checkerboard.m}}$

Conservative finite differences / Finite Volumes

To reduce width of FD stencil and couple even & odd terms we can introduce a staggered grid comprising control volumes





The head is approximated in the center of the control volume and fluxes are approximated on the faces of the control volume.

Discretize the div-grad system:

1) $\nabla \cdot q = f \xrightarrow{1D} \frac{dq}{dx} = f \xrightarrow{CFD} \frac{q_{i+1} - q_i}{\Delta x} = f_i$

2) $q = -K \nabla h \xrightarrow{1D} q = -K \frac{dh}{dx} \xrightarrow{CFD} q_i = -K_{i+\frac{1}{2}} \frac{h_i - h_{i-1}}{\Delta x}$

Note K is usually given in cell centers
$\Rightarrow$ appropriate average on cell interfaces

substitute ② into ① to obtain equ for h only

$$-\frac{1}{\Delta x}\left[K_{i+\frac{1}{2}} \frac{h_{i+1} - h_i}{\Delta x} - K_{i-\frac{1}{2}} \frac{h_i - h_{i-1}}{\Delta x}\right] = f_i$$

$$-\frac{1}{\Delta x^2}\left[K_{i+\frac{1}{2}} h_{i+1} - \left(K_{i+\frac{1}{2}} + K_{i-\frac{1}{2}}\right) h_i + K_{i-\frac{1}{2}} h_{i-1}\right] = f_i$$

$\Rightarrow$ now we have a narrow stencil