Darcy - Stokes Equations

We have discussed the flow of a porefluid (Darcy flow) and the flow of a very viscous fluid (Stokes flow) and a simplified model for melt migration in a viscous matrix.

We now derive the full Darcy-Stokes equation for melt-migration in a viscous matrix experiencing large deformations.

We assume 2 phases: pore fluid (f) → melt

solid matrix (s) → ice

Mass conservation:

fluid: $\boxed{\frac{\partial}{\partial t}(\phi \rho_f) + \nabla\cdot[\phi \underline{v}_f \rho_f] = \Gamma}$

solid: $\boxed{\frac{\partial}{\partial t}[(1-\phi)\rho_s] + \nabla\cdot[(1-\phi)\underline{v}_s \rho_s] = -\Gamma}$

$\phi = \phi_f$ is the porosity which is equal to the fluid volume fraction ($\Rightarrow$ saturated porous medium)

$1-\phi = \phi_s$ solid volume fraction

ρ_f, ρ_s are fluid & solid densities assumed to be constant but different

Dividing by the densities and summing we obtain the total mass balance eqn:

$$\nabla\cdot[\phi \underline{v}_f + (1-\phi)\underline{v}_s] = \frac{\Gamma}{\rho_f} - \frac{\Gamma}{\rho_s} = \frac{\rho_s-\rho_f}{\rho_s\rho_f}\Gamma$$

introduce: $\Delta\rho = \rho_f - \rho_s > 0$

Two-phase continuity:

$$\boxed{\begin{gathered}\nabla\cdot[\phi \underline{v}_f + (1-\phi)\underline{v}_s] = -\frac{\Delta\rho}{\rho_f\rho_s}\Gamma \\ \nabla\cdot[\underline{q}_r + \underline{v}_s] = -\frac{\Delta\rho}{\rho_f\rho_s}\Gamma\end{gathered}}$$

Linear momentum conservation

fluid: $\nabla\cdot[\phi\,\underline{\underline{\sigma}}_f] - \phi\rho_f g\hat{z} - \underline{f}_I = \underline{0}$

solid: $\nabla\cdot[(1-\phi)\underline{\underline{\sigma}}_s] - (1-\phi)\rho_s g\hat{z} + \underline{f}_I = \underline{0}$

$\underline{\underline{\sigma}}_f$ = fluid stress tensor

$\underline{\underline{\sigma}}_s$ = solid stress tensor

$\hat{z} = \nabla z$ where z points upward

$\underline{f}_I$ = interaction force between solid and fluid

Summing we obtain the total momentum eqn.:

$$\nabla\cdot[\phi\,\underline{\underline{\sigma}}_f + (1-\phi)\underline{\underline{\sigma}}_s] - [\phi\rho_f + (1-\phi)\rho_s]g\hat{z} = \underline{0}$$

introducing: $\bar{\rho} = \phi\rho_f + (1-\phi)\rho_s$

we have

$$\boxed{\nabla\cdot[\phi\,\underline{\underline{\sigma}}_f + (1-\phi)\underline{\underline{\sigma}}_s] - \bar{\rho} g\hat{z} = 0}$$

Viscous stress tensor

Spherical-Deviatoric Decomposition

Any second rank tensor $\underline{\underline{A}}$ can be decomposed

$$\underline{\underline{A}} = \alpha\,\underline{\underline{I}} + \mathrm{dev}(A)$$

Spherical tensor : $\alpha\,\underline{\underline{I}} = \frac{1}{3}\,\mathrm{tr}(\underline{\underline{A}})\,\underline{\underline{I}}$

Deviatoric tensor: $\mathrm{dev}(\underline{\underline{A}}) = \underline{\underline{A}} - \alpha\,\underline{\underline{I}}$

by definition $\mathrm{tr}(\mathrm{dev}(\underline{\underline{A}})) = 0$

Applied to Cauchy stress tensor:

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + \underline{\underline{\tau}}$$

$p = -\frac{1}{3}\,\mathrm{tr}(\underline{\underline{\sigma}})$ "mean isotropic stress" = pressure

$\underline{\underline{\tau}} = \underline{\underline{\sigma}} - \frac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}})\,\underline{\underline{I}} = \underline{\underline{\sigma}} + p\,\underline{\underline{I}}$ deviatoric stress

Newtonian fluid: $\boxed{\underline{\underline{\tau}} = 2\mu\,\dot{\underline{\underline{\varepsilon}}}}$

$\dot{\underline{\underline{\varepsilon}}}$ = deviatoric rate of strain tensor

$\dot{\underline{\underline{e}}} = \frac{1}{2}\left(\nabla\underline{v} + \nabla^{T}\underline{v}\right)$ full rate of strain tensor

$\Rightarrow\ \dot{\underline{\underline{\varepsilon}}} = \dot{\underline{\underline{e}}} - \frac{1}{3}\,\mathrm{tr}(\dot{\underline{\underline{e}}})\,\underline{\underline{I}}$

What is $\mathrm{tr}(\dot{\underline{\underline{e}}})$?

$\dot{e}_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$

$tr(\underline{\underline{\dot{e}}}) = \dot{e}_{ii} = \frac{1}{2}(v_{i,i} + v_{i,i}) = v_{i,i} = \nabla \cdot \underline{v}$

Deviatoric strain rate tensor:

$\underline{\underline{\dot{\varepsilon}}} = \underline{\underline{\dot{e}}} - \frac{1}{3}(\dot{e})\underline{\underline{I}} = \frac{1}{2}(\nabla\underline{v} + \nabla^T\underline{v}) - \frac{1}{3}\nabla\cdot\underline{v}\,\underline{\underline{I}}$

substitute into $\underline{\underline{\tau}}$:

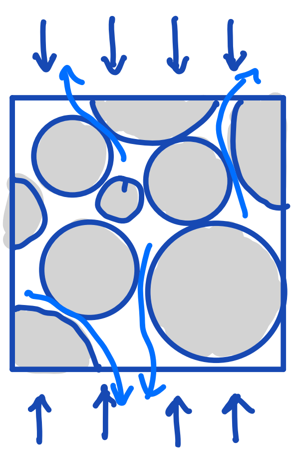
$\underline{\underline{\tau}} = 2\mu\,\underline{\underline{\dot{\varepsilon}}} = \mu\left(\nabla\underline{v} + \nabla^T\underline{v} - \frac{2}{3}\nabla\cdot\underline{v}\,\underline{\underline{I}}\right)$

Newtonian deviatoric stress

$$\boxed{\underline{\underline{\tau}} = \mu\left(\nabla\underline{v} + \nabla^T\underline{v} - \frac{2}{3}\nabla\cdot\underline{v}\,\underline{\underline{I}}\right)}$$

So far we have considered incompressible cases where $\nabla\cdot\underline{v} = 0$ by continuity

$\Rightarrow \quad \underline{\underline{\tau}} = \mu(\nabla\underline{v} + \nabla^T\underline{v})$



Ice (solid) is incompressible but ice + melt mixture is compressible!

$\Rightarrow \nabla\cdot\underline{v}_s \neq 0$ (two-phase continuity)

Compressible Newtonian Fluid

General compressible Cauchy stress tensor

$$\boxed{\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + \lambda\, \nabla\cdot\underline{v}\,\underline{\underline{I}} + \mu\left(\nabla\underline{v} + \nabla^T\underline{v}\right)}$$

p = thermodynamic pressure (eqbm) $\rho = \rho(p)$

μ = shear viscosity

λ = second viscosity (related to compression)

Mechanical pressure:

$$p_m = -\tfrac{1}{3}\operatorname{tr}(\underline{\underline{\sigma}}) = -\tfrac{1}{3}\operatorname{tr}\left[-p\underline{\underline{I}} + \lambda(\nabla\cdot\underline{v}) + \mu(\nabla v + \nabla^T v)\right]$$

$$= p - \left(\lambda + \tfrac{2}{3}\mu\right)\nabla\cdot\underline{v}$$

$$\boxed{p_m = \underbrace{p}_{\text{thermo. pres.}} - \underbrace{\zeta\,\nabla\cdot\underline{v}}_{\text{dyn. pres.}}}$$

$\boxed{\zeta = \lambda + \tfrac{2}{3}\mu}$ bulk viscosity

If flow is not divergence free the mechanical pressure differs from eqbm thermodynamic pressure. Diverging flows have a lower mechanical pressure!

Two pressures are the same $p = p_m$ if

either $\nabla\cdot\underline{v} = 0$ or $\varsigma = 0$

Rewrite Cauchy stress in terms of ς:

$$\boxed{\begin{aligned}\underline{\underline{\sigma}} &= -(p - \varsigma\,\nabla\cdot\underline{v})\underline{\underline{I}} + \mu\left(\nabla\underline{v} + \nabla^T\underline{v} - \tfrac{2}{3}\nabla\cdot\underline{v}\,\underline{\underline{I}}\right)\\ \underline{\underline{\sigma}} &= -p_m\underline{\underline{I}} + \underline{\underline{\tau}}\end{aligned}}$$

linear momentum balance:

$$-\nabla\cdot\underline{\underline{\sigma}} + \rho g\hat{z} = 0$$

$$-\nabla\cdot(\underline{\underline{\tau}} - p_m\underline{\underline{I}}) + \rho g\hat{z} = -\nabla\cdot\underline{\underline{\tau}} + \nabla p_m + \rho g\hat{z} = 0$$

so that final lin. momentum balance is

$$\boxed{-\nabla\cdot\left[\mu\left(\nabla\underline{v} + \nabla^T\underline{v} - \tfrac{2}{3}\nabla\cdot\underline{v}\,\underline{\underline{I}}\right)\right] + \nabla(p - \varsigma\,\nabla\cdot\underline{v}) + \rho g\hat{z} = 0}$$

The mass balance in a compressible flow is

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\underline{v}\rho) = 0}$$

Note: In real compressible flows it is typically necessary to consider energy equation as well.

Total stress for Darcy-Stokes

Total momentum balance:

$$\nabla \cdot [\phi \underline{\underline{\sigma}}_f + (1-\phi) \underline{\underline{\sigma}}_s] = \bar{\rho} g \hat{z}$$

Need to define $\underline{\underline{\sigma}}_f$ and $\underline{\underline{\sigma}}_s$

1) Stress in the pore fluid

The pore fluid does not accomodate deviatoric stress, $\underline{\underline{\tau}}_f = \underline{\underline{0}}$.

$$\boxed{\underline{\underline{\sigma}}_f = -p_f \underline{\underline{I}}} \qquad p_f \text{ - fluid pressure}$$

Show that this stress reduces lin. momentum balance to Darcy's law: $\underline{q}_r = \phi(\underline{v}_f - \underline{v}_s) = -\frac{k}{\mu}(\nabla p_f + \rho_f g \hat{z})$

Start with lin. momentum balance in fluid:

$$\nabla \cdot [\phi \underline{\underline{\sigma}}_f] - \phi \rho_f g \hat{z} - \underline{f}_I = \underline{0}$$

Need expression for the interaction force

$$\boxed{\underline{f}_I = c(\underline{v}_f - \underline{v}_s) - p_I \nabla \phi} \qquad p_I = \text{interface pressure}$$

Simplest expression with Galilean invariance.

The first term in $\underline{f}_I$ is the viscous interaction between the phases, i.e. the drag force if they move with different velocities.

The second term in $\underline{f}_I$ is due to the pressure acting on the interface. It allows for no motion if the fluid pressure is hydrostatic.

Note: Authors differ on how to choose p_I

McKenzie (1984) : $p_I = p_f$

Bercovici et al (2001) : $p_I = (1-\phi) p_f + \phi p_s$

Here we follow the classic version of McKenzie.

Now we have the following 3 relations:

lin. mom. bal.: $\nabla \cdot [\phi \underline{\underline{\sigma}}_f] - \phi \rho_f g \hat{z} - \underline{f}_I = \underline{0}$

Cauchy stress: $\underline{\underline{\sigma}}_f = -p_f \underline{\underline{I}}$

Interphase force: $\underline{f}_I = c(\underline{v}_f - \underline{v}_s) - p_f \nabla \phi$

Substituting we have

$$-\nabla\cdot[\phi p_f \underline{\underline{I}}] - \phi\rho_f g\hat{z} - c(\underline{v}_f - \underline{v}_s) + p_f\nabla\phi = 0$$

$$-\phi\nabla p_f - \cancel{p_f\nabla}\phi - \phi\rho_f g\hat{z} - c(\underline{v}_f - \underline{v}_s) + \cancel{p_f\nabla}\phi = 0$$

$$c(\underline{v}_f - \underline{v}_s) = -\phi(\nabla p_f + \rho_f g\hat{z})$$

compare with Darcy's law to find c:

Darcy: $\phi(\underline{v}_f - \underline{v}_s) = -\frac{k}{\mu}(\nabla p_f + \rho_f g\hat{z})$

$$\Rightarrow \quad \boxed{c = \frac{\phi^2\mu}{k}}$$

Hence we have shown that

$$\underline{\underline{\sigma}}_f = -p_f\underline{\underline{I}}$$

$$\underline{f}_I = \frac{\phi^2\mu_f}{k}(\underline{v}_f - \underline{v}_s) - p_f\nabla\phi$$

reduce lin. momentum balance to Darcy's law.

2) Stress in viscous matrix

General Newtonian stress tensor:

$$\underline{\underline{\sigma}}_s = -p_s \underline{\underline{I}} + \lambda_s \nabla\cdot\underline{v}_s \underline{\underline{I}} + \mu_s (\nabla\underline{v}_s + \nabla\underline{v}_s^T)$$

The viscous solid is incompressible $\lambda_s = 0$

but $\nabla\cdot\underline{v}_s \neq 0$ due to compaction.

$$\underline{\underline{\sigma}}_s = -p_s \underline{\underline{I}} + \underline{\underline{\tau}}_s \qquad p_s = \text{solid pressure}$$

$$= -p_s \underline{\underline{I}} + \mu_s\left(\nabla\underline{v}_s + \nabla\underline{v}_s^T - \tfrac{2}{3}\nabla\cdot\underline{v}_s \underline{\underline{I}}\right)$$

Substitute into total momentum balance:

$$\nabla\cdot\left[\phi\, \underline{\underline{\sigma}}_f + (1-\phi)\, \underline{\underline{\sigma}}_s\right] = \bar{\rho} g \hat{z}$$

$$\nabla\cdot\left[-\phi p_f \underline{\underline{I}} - (1-\phi) p_s \underline{\underline{I}} + (1-\phi)\underline{\underline{\tau}}_s\right] = \bar{\rho} g \hat{z}$$

$$\nabla\cdot\left[-(\phi p_f + (1-\phi) p_s)\underline{\underline{I}} + (1-\phi)\underline{\underline{\tau}}_s\right] = \bar{\rho} g \hat{z}$$

Introduce compaction relation (as before)

$$\boxed{p_f - p_s = \frac{G\mu_s}{\phi^m}\nabla\cdot\underline{v}_s}$$

here G is a constant $\mathcal{O}(1)$

$\Rightarrow$ eliminate solid pressure $\quad p_s = p_f - \frac{G\mu_s}{\phi^m}\nabla\cdot\underline{v}_s$

$$\nabla \cdot \left[-\left\langle \phi p_f + (1-\phi)\left(p_f - \frac{G \mu_s}{\phi^m} \nabla \cdot \underline{v}_s\right)\right\rangle \underline{\underline{I}} + (1-\phi)\, \underline{\underline{\tau}}_s \right] = \bar{\rho} g \hat{\underline{z}}$$

$$\nabla \cdot \left[- p_f \underline{\underline{I}} + \frac{1-\phi}{\phi^m} G \mu_s \nabla \cdot \underline{v}_s \underline{\underline{I}} + (1-\phi)\, \underline{\underline{\tau}}_s \right] = \bar{\rho} g \hat{\underline{z}}$$

$$\nabla \cdot (-p_f \underline{\underline{I}}) = -\nabla p_f \qquad \boxed{\xi_e = \frac{1-\phi}{\phi^m} \mu_s G}$$

$$-\nabla (p_f - \xi_e \nabla \cdot \underline{v}_s) + \nabla \cdot [(1-\phi)\, \underline{\underline{\tau}}_s] = \bar{\rho} g \hat{\underline{z}}$$

Substitute $\underline{\underline{\tau}}_s$ with $\mu_s^* = (1-\phi\, \mu_s)$

Total momentum balance:

1) $\boxed{-\nabla (p_f - \xi_e \nabla \cdot \underline{v}_s) + \nabla \cdot \left[\mu_s^* \left(\nabla \underline{v}_s + \nabla^T \underline{v}_s - \frac{2}{3} \nabla \cdot \underline{v}_s \underline{\underline{I}}\right)\right] = \bar{\rho} g \hat{z}}$

Same form as single phase compressible flow eqn.

Total mass balance

2) $\boxed{\nabla \cdot [\underline{q}_r + \underline{v}_s] = - \frac{\Delta \rho}{\rho_f \rho_s} \Gamma}$

Two constitutive laws:

3) Darcy: $\underline{q}_r = - \frac{k_\phi}{\mu_f} (\nabla p_f + \rho_f g \hat{z})$

4) Compaction: $p_f - p_s = \frac{G}{\phi^m} \nabla \cdot \underline{v}_s$

$\Rightarrow$ 4 equations for 4 unknowns: $p_s, p_f, \underline{v}_s, \underline{v}_f$