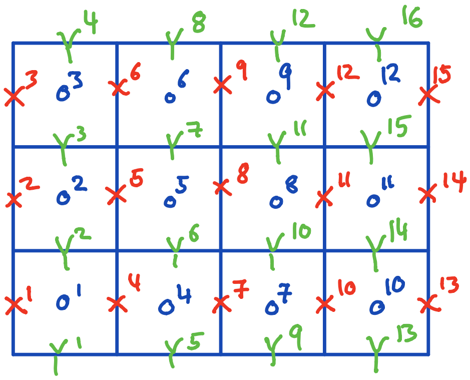
Discrete operators in 2D

Staggered grid in 2D



$N_x = 4,\ N_y = 3,\ \Rightarrow N = N_x N_y = 12$

faces in x-dir.: $N_{fx} = (N_x + 1) N_y = 15$

faces in y-dir.: $N_{fy} = N_x (N_y + 1) = 16$

Total faces : $N_f = N_{fx} + N_{fy} = 31$

Discrete gradient in 2D:

Continuous gradient: $\nabla h = \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{pmatrix}$

approximate $\frac{\partial h}{\partial x} \sim \underline{dhx}$ on x-faces

approximate $\frac{\partial h}{\partial y} \sim dhy$ on y-faces

Choose to build $\underline{\underline{G}}$ such that the resulting gradient vector is ordered as $\underline{dh} = \begin{bmatrix} \underline{dhx} \\ \underline{dhy} \end{bmatrix}$

$\Rightarrow$ 2D Gradient can be decomposed as

$$\underline{\underline{G}} = \begin{bmatrix} \underline{\underline{Gx}} \\ \underline{\underline{Gy}} \end{bmatrix} \qquad \text{where} \qquad \begin{aligned} \underline{dhx} &= \underline{\underline{Gx}}\ \underline{h} \\ \underline{dhy} &= \underline{\underline{Gy}}\ \underline{h} \end{aligned}$$

$$\begin{array}{r} N_{fx}\, \{ \\ N_{fy}\, \{ \end{array} \overset{\underline{dh}}{\begin{bmatrix} \underline{dhx} \\ \underline{dhy} \end{bmatrix}} = \overbrace{\begin{bmatrix} \underline{\underline{Gx}} \\ \underline{\underline{Gy}} \end{bmatrix}}^{N} \left.\begin{bmatrix} \underline{h} \end{bmatrix}\right\} N$$

Matrix dimensions:

$\underline{\underline{G}}$ is $Nf \times N$

$\underline{\underline{Gx}}$ is $Nfx \times N$

$\underline{\underline{Gy}}$ is $Nfy \times N$

Discrete divergence in 2D

$$\nabla \cdot \underline{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \approx \underline{\underline{D}}\,\underline{q} = \underline{\underline{Dx}}\,\underline{qx} + \underline{\underline{Dy}}\,\underline{qy}$$

$$N \overset{\underline{fs}}{\begin{bmatrix} \; \end{bmatrix}} = \begin{bmatrix} \overbrace{\underline{\underline{Dx}}}^{Nfx} & \overbrace{\underline{\underline{Dy}}}^{Nfy} \end{bmatrix} \overset{\underline{q}}{\begin{bmatrix} \underline{qx} \\ \underline{qy} \end{bmatrix}} \begin{array}{l} \} Nfx \\ \} Nfy \end{array}$$

matrix sizes:

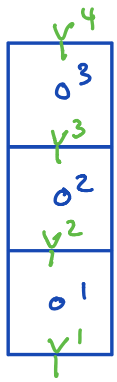
$\underline{\underline{D}}$ is N by Nf

$\underline{\underline{Dx}}$ is N by Nfx

$\underline{\underline{Dy}}$ is N by Nfy

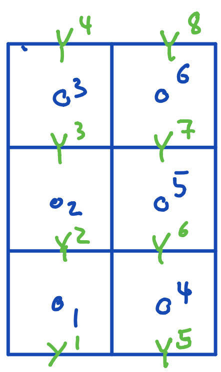
Building the 2D discrete divergence matrix

Start with $\underline{\underline{D_y}}$ in 1D:



$$\underset{\underline{f_s}}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} = \frac{1}{\Delta y} \underset{N_y \times (N_y+1)}{\overset{\underline{\underline{D_y^1}}}{\begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix}}} \underset{\underline{q_y}}{\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}}$$

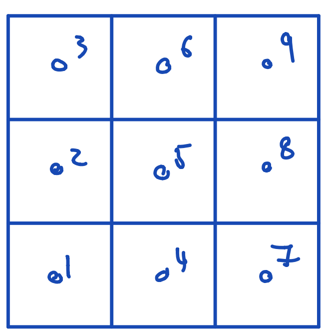
Suppose we add a second column of cells



$$\underline{f_s}\ N_y\left\{\begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}\right. = \frac{1}{\Delta y} \overset{\underline{\underline{D_y^2}}}{\begin{bmatrix} -1 & 1 & & & & & & \\ & -1 & 1 & & & & & \\ & & -1 & 1 & & & & \\ & & & & -1 & 1 & & \\ & & & & & -1 & 1 & \\ & & & & & & -1 & 1 \end{bmatrix}} \left.\underline{q_y}\begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}\right\} N_y+1$$

$$\underline{\underline{D_y^2}} = \begin{bmatrix} \underline{\underline{D_y^1}} & \\ & \underline{\underline{D_y^1}} \end{bmatrix}$$

2 by 2 block matrix with $\underline{\underline{D_y^1}}$ on diagonal



$$\underline{\underline{D_y^2}} = \begin{bmatrix} \underline{\underline{D_y^1}} & & \\ & \underline{\underline{D_y^1}} & \\ & & \underline{\underline{D_y^1}} \end{bmatrix}$$

In general:

$\underline{\underline{D_y^2}}$ is a block matrix with N_x by N_x blocks of size N_y by (N_y+1). Diagonal blocks are $\underline{\underline{D_y^1}}$ and all others are zero.

Tensor product construction of $\underline{\underline{D_y^2}}$

The discrete 2D operator can easily and efficiently be assembled using Konecker/tensor products.

Definition:

If $\underline{\underline{A}}$ is a $m \times n$ matrix and $\underline{\underline{B}}$ is a $p \times q$ matrix, then the Kronecker product $\underline{\underline{A}} \otimes \underline{\underline{B}}$ is the $mp \times nq$ block matrix:

$$\underline{\underline{A}} \otimes \underline{\underline{B}} = \begin{bmatrix} a_{11}\underline{\underline{B}} & \cdots & a_{1n}\underline{\underline{B}} \\ \vdots & \ddots & \vdots \\ a_{m1}\underline{\underline{B}} & \cdots & a_{mn}\underline{\underline{B}} \end{bmatrix}$$

Hence we can construct D_y^2 as

$$\boxed{D_y^2 = I_x \otimes D_y^1} = \begin{bmatrix} D_y^1 & & & & \\ & D_y^1 & & & \\ & & D_y^1 & & \\ & & & \ddots & \\ & & & & D_y^1 \end{bmatrix}$$

where I_x is a N_x by N_x identity matrix.

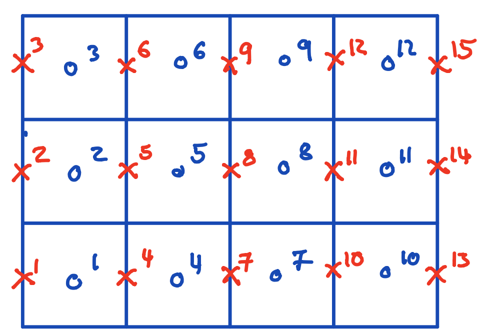
In Matlab the tensor product is obtained as

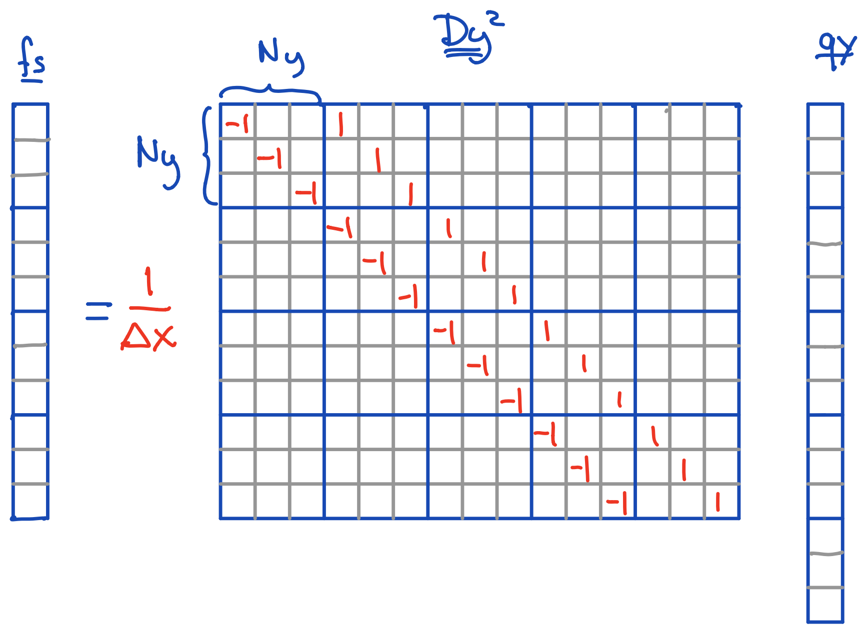
$$\underset{\substack{\uparrow \\ \text{2D op.}}}{D_y} = \text{kron}(I_x, \underset{\substack{\uparrow \\ \text{1D op}}}{D_y});$$

So how do we build D_x^2?

On a x-first grid $D_x^2 = I_y \otimes D_x^1$!

But what does D_x^2 look like on a y-first grid?





$\Rightarrow$ $\underline{\underline{Dx}}^2$ is a sparse diagonal matrix

(this could be assembled with spdiags)

$\underline{\underline{Dx}}^2$ is also a block matrix built from

Ny by Ny Identities matrices.

$$\underline{\underline{Dx}}^2 = \begin{bmatrix} -\underline{\underline{I_y}} & \underline{\underline{I_y}} & & & & \\ & -\underline{\underline{I_y}} & \underline{\underline{I_y}} & & & \\ & & -\underline{\underline{I_y}} & \underline{\underline{I_y}} & & \\ & & & -\underline{\underline{I_y}} & \underline{\underline{I_y}} & \\ & & & & -\underline{\underline{I_y}} & \underline{\underline{I_y}} \end{bmatrix} = \underline{\underline{Dx}}' \otimes \underline{\underline{I_y}}$$

In Matlab: $\underline{\underline{Dx}}$ = kron($\underline{\underline{Dx}}$, $\underline{\underline{I_y}}$)

Discrete gradient matrix

The $\underline{\underline{G}}x$ and $\underline{\underline{G}}y$ matrices could be built using 1D matrices and Kronecker products.

Instead, we use the fact that the $\underline{\underline{D}}$ and $\underline{\underline{G}}$ matrices are adjoints:

$$\boxed{\underline{\underline{G}} = -\underline{\underline{D}}^T} \quad \text{true in interior}$$

Need to impose natural BC's. $\Rightarrow$ set $\underline{\underline{G}} = 0$ on all boundary faces.

Make vector containing all bnd faces:

```
dof_f_bnd = [dof_f_xmin; dof_f_xmax; ...
             dof_f_ymin; dof_f_ymax];
```

Zero out corresponding rows in $\underline{\underline{G}}$:

```
G(dof_f_bnd, :) = 0;
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