Fluid & Solid Mass Balances

General balance eqn.: $\frac{\partial u}{\partial t} + \nabla \cdot \vec{j}(u) = \hat{f}_s(u)$

I. Fluid mass balance

1) Unknown to be balanced: $u = \phi \rho_f$

   "fluid mass per unit volume of porous medium"

2) Define mass flux of pore fluid

   $\vec{j}(u) = \vec{j}(\phi\rho_f) = \rho_f \phi \bar{v}_f = \rho_f \bar{q}_f$

3) Source term: $\hat{f}_s = \rho_f \Gamma$ $\qquad \Gamma$ = vol. rate of melting/freezing

Fluid mass balance: $\boxed{\frac{\partial}{\partial t}(\phi\rho_f) + \nabla \cdot (\rho_f \bar{q}_f) = \rho_f \Gamma}$

II. Solid mass balance

1) unknown to be balanced: $u = (1-\phi)\rho_s$

2) mass flux of solid: $\vec{j}((1-\phi)\rho_s) = \rho_s (1-\phi) \vec{v}_s$

3) Source term: $\hat{f}_s = -\rho_s \Gamma$

Solid mass balance: $\boxed{\frac{\partial}{\partial t}\left((1-\phi)\rho_s\right) + \nabla \cdot \left((1-\phi)\rho_s \bar{v}_s\right) = -\rho_s \Gamma}$

For now we assume no phase change: $\Gamma = 0$

I.) $\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \bar{v}_f) = 0$

II.) $-\frac{\partial \phi}{\partial t} + \nabla \cdot \left((1-\phi)\vec{v}_s\right) = 0$

Two-phase continuity equation:

$$\boxed{\nabla \cdot (\phi \bar{v}_f + (1-\phi)\bar{v}_s) = \nabla \cdot (\bar{q}_r + \bar{v}_s) = 0}$$

$\bar{q}_r$ can be eliminated using Darcy's law

Need an additional constitutive law for the volumetric strain rate $\dot{\varepsilon} = \nabla \cdot \bar{v}_s$

1) Elastic matrix (standard case):

$$\nabla \cdot \bar{v}_s \equiv c_r \frac{\partial p_f}{\partial t}$$

c_r = bulk rock compressibility $\left[\frac{LT^2}{M}\right]$

$$c_r = -\frac{1}{V_T} \frac{dV_T}{d\sigma'}\bigg|_T$$

$V_T = V_f + V_s$ total rock volume

σ' = effective stress, i.e. stress on the solid

$\sigma_T = \sigma' + p_f$ $\quad \sigma_T$ = total stress

Substituting into continuity

$$\nabla \cdot (\bar{q}_r + \bar{v}_s) = \nabla \cdot \bar{v}_s + \nabla \cdot \bar{q}_r = c_r \frac{\partial p_f}{\partial t} - \nabla \cdot \left(\frac{k}{\mu}(\nabla p_f + \rho_f g \hat{z})\right) = 0$$

Standard groundwater flow equation:

$$\boxed{c_r \frac{\partial p_f}{\partial t} - \nabla \cdot \left(\frac{k}{\mu}(\nabla p_f + \rho_f g \hat{z})\right) = 0}$$ equation for p_f

impled porosity change: $\phi \sim \phi_0 e^{c_r(p-p_0)}$

typically $c_r \approx 10^{-8} \frac{1}{Pa}$

Δp due to 100 m water column $\Delta p = \rho_f g h = 10^3 \frac{kg}{m^3}\, 10^1 \frac{m}{s^2}\, 10^2 m = 10^6$ Pa

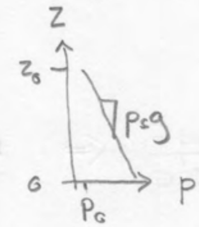
$$\Delta \phi \sim \phi_0 e^{10^{-2}} \approx \phi_0$$

$\Rightarrow$ porosity change is negligible

2) Ductile/viscous matrix

$p_f - p_s = \zeta\, \nabla \cdot \bar{v}_s$ $\qquad$ ζ = bulk viscosity of two-phase medium

$p = p_f - p_s$ $\quad$ overpressure in the fluid



Assume that $\quad p_s = p_0 + \rho_s g (z_0 - z)$

Reformulate Darcy's law in terms of over pressure

$$q_r = -\frac{k}{\mu}\left(\nabla p_f + \rho_f g \hat{z}\right) = -\frac{k}{\mu}\Big(\underbrace{\nabla p_f - \nabla p_s}_{\nabla p} + \underbrace{\nabla p_s}_{-\rho_s g \hat{z}} + \rho_f g \hat{z}\Big)$$

$$\boxed{q_r = -\frac{k}{\mu}\left(\nabla p + \Delta\rho g \hat{z}\right)} \qquad \Delta\rho = \rho_f - \rho_s$$

Substitute into two phase continuity equation:

$$\nabla \cdot (\bar{q}_r + \bar{v}_s) = \nabla \cdot \bar{q}_r + \nabla \cdot \bar{v}_s = 0$$

$$= -\nabla \cdot \left(\frac{k}{\mu}(\nabla p + \Delta\rho g \hat{z})\right) + \frac{p}{\zeta} = 0$$

Compaction equation: $\boxed{-\nabla \cdot \left(\frac{k}{\mu}(\nabla p + \Delta\rho g \hat{z})\right) + \frac{p}{\zeta} = 0}$

Needs to be coupled to a porosity evolution equation

From solid mass balance:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \bar{v}_s) = \nabla \cdot v_s \quad \Rightarrow \quad \boxed{\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \bar{v}_s) = \frac{p}{\zeta}}$$

In the ductile case we have to consider the variation of k and ζ with ϕ.

Typically:

$$k = k_0 \phi^n \qquad n \in [2,3]$$

$$\zeta = \frac{\zeta_0}{\phi^m} \qquad m \in [0,1]$$