# Lecture 10: Melt migration

Logistics: - HW3 is posted

Last time: - Neumann/Flux BC's

convert flux to source term: $f_n = q_b \frac{A}{V}$

$\underline{\underline{L}}\,\underline{h} = \underline{f}_s + \underline{f}_n$

- Flux computation

interior: $\underline{q} = -\underline{\underline{Kd}}\,\underline{\underline{G}}\,\underline{h}$

boundary: $q_b = r\frac{V}{A}$

Summary numerics:

- build_grid
- build_ops
- comp_mean
- build_bnd
- solve_lbvp
- comp_flux

} updated as we go along

⇒ solve steady 1D problems

Today: Melt migration

- derive the governing equations
- head-based formulation
- non-dimensionalize

## Intro to melt migration

So far rigid rock $\rightarrow$ $\underline{v}_s = 0$

Class project $\rightarrow$ partially molten ice

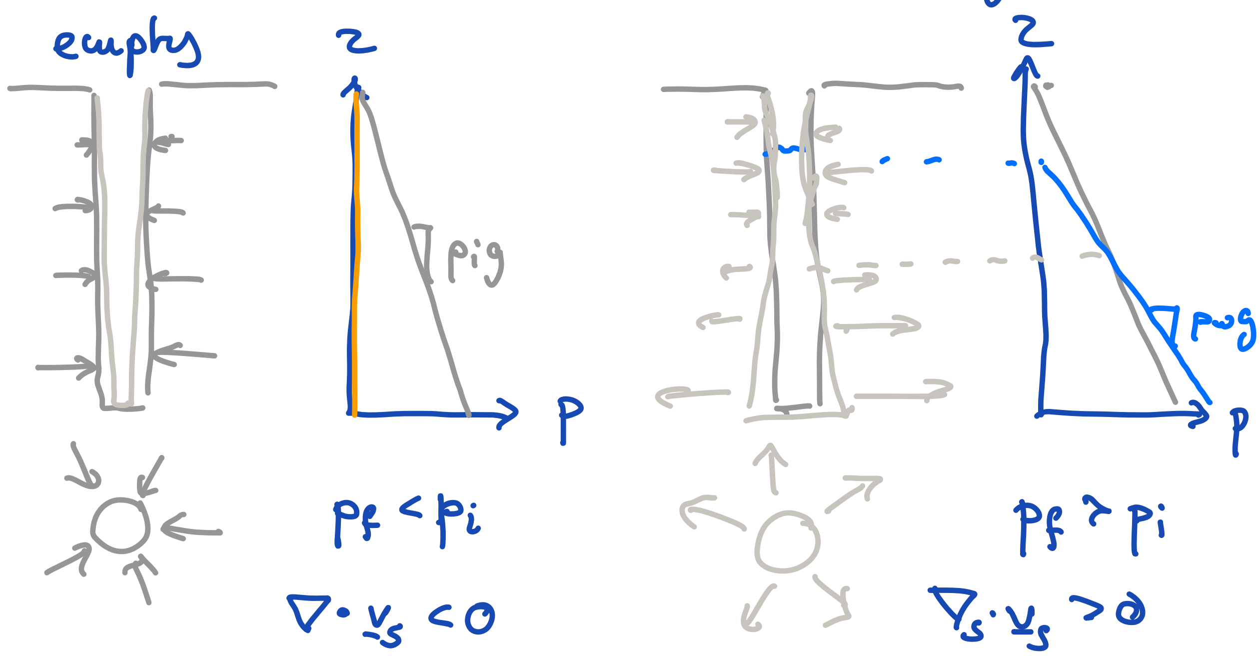
Ice (matrix) is not rigid and deforms by ductile creep. Simplest model ~~is~~ for creep is a very viscous fluid.

How viscous? water 1 Pas

ice $10^{12} - 10^{14}$ Pas

Key feature of viscous rheology is that cannot support any shear stress

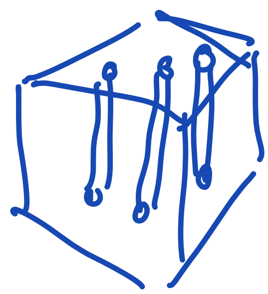
Consider a bore hole in ice (Nye 1953)




$p_f < p_i$ $\qquad$ $p_f > p_i$

$\nabla \cdot \underline{v}_s < 0$ $\qquad$ $\nabla_s \cdot \underline{v}_s > 0$

Translate idea to porous medium of ice by imagining a block with a set of small tubes



This leads to compaction relation

$$\boxed{p_f - p_s = \xi \underbrace{\nabla \cdot \underline{v}_s}_{\dot{\varepsilon}_{vol}}}$$

$\xi > 0$ bulk or compaction viscosity

Empirical law similar to Darcy's law

Compaction viscosity: $\boxed{\xi = c \dfrac{\eta}{\phi^m}}$

$\eta$ = shear viscosity

$m$ = exp   0 - 1

$c$ = coeff of proportionality   $c \sim 1$

Overpressure in fluid   $p = p_f - p_s$

Assume: $p_s = p_0 + \rho_s g (z_0 - z)$

solid pressure is lithostatic

$$\nabla p_s = -\rho_s g \hat{z}$$

Reformulate Darcy in terms of overpressure

head: $q_r = -K \nabla h$   $K$ = hyd. cond.   $h$ = head

$$q_r = -\frac{k}{\mu_f} \left( \nabla p_f + \rho_f g \hat{z} \right)$$

$\hat{z}$ points upward

$$\boxed{k = k_0 \phi^n}$$

$$= -\frac{k}{\mu_f} \left( \nabla p_f - \nabla p_s + \nabla p_s + \rho_f g \hat{z} \right)$$

$$= -\frac{k}{\mu_f} \left( \nabla (\underbrace{p_f - p_s}_{p}) - \rho_s g \hat{z} + \rho_f g \hat{z} \right)$$

$\Delta \rho = \rho_f - \rho_s > 0$

$$\boxed{q_r = -\frac{k}{\mu_f} \left( \nabla p + \Delta \rho g \hat{z} \right)}$$

# Mass balance eqns

fluid: $\frac{\partial}{\partial t}(\rho_f \phi) + \nabla \cdot [\rho_f \phi \underline{v}_f] = \Gamma$

solid: $\frac{\partial}{\partial t}(\rho_s (1-\phi)) + \nabla \cdot [\rho_s (1-\phi) \underline{v}_s] = -\Gamma$

where $\Gamma$ is melting/freezing rate $\frac{M}{L^3 T}$

Assume: $\rho_f$ = const $\rho_s$ = const

$$\frac{\partial \phi}{\partial t} + \nabla \cdot [\phi \underline{v}_f] = \frac{\Gamma}{\rho_f}$$

$$-\frac{\partial \phi}{\partial t} + \nabla \cdot [(1-\phi) \underline{v}_s] = -\frac{\Gamma}{\rho_s}$$

sum both $\rightarrow$ continuity equation

$$\nabla \cdot [\phi \underline{v}_f + \underset{T}{(1-\phi)} \underline{v}_s] = \frac{\Gamma}{\rho_f} - \frac{\Gamma}{\rho_s} = \frac{\rho_s - \rho_f}{\rho_f \rho_s} \Gamma = -\frac{\Delta \rho}{\rho_f \rho_s} \Gamma$$

Darcy's law: $q_r = \phi (\underline{v}_f - \underline{v}_s)$

$$\nabla \cdot [\underbrace{\phi (\underline{v}_f - \underline{v}_s)}_{q_r} + \underline{v}_s] = -\frac{\Delta \rho}{\rho_f \rho_s} \Gamma$$

Two-phase continuity eqn:

$$\boxed{\nabla \cdot [q_r + \underline{v}_s] = -\frac{\Delta \rho}{\rho_f \rho_s} \Gamma}$$

Substitute two constitutive laws:

1) Darcy: $\underline{q} = -\frac{k}{\mu_f}(\nabla p + \Delta\rho g \hat{z})$

2 Comp: $p = \xi \nabla \cdot \underline{v}_s$

so that we have

$$-\nabla \cdot \left[\frac{k}{\mu_f}(\nabla p + \Delta\rho g \hat{z})\right] + \frac{p}{\xi} = -\frac{\Delta\rho}{\rho_f \rho_s}\Gamma$$

simplify by introducing overpressure head

$$h = z + \frac{p}{\Delta\rho g} \Rightarrow p = \Delta\rho g (h - z)$$

$$\nabla p = \Delta\rho g (\nabla h - \hat{z})$$

substitute into Darcy

$$q_r = -\frac{k}{\mu_f}(\nabla p + \Delta\rho g \hat{z})$$

$$= -\frac{k}{\mu_f}\left(\Delta\rho g(\nabla h - \cancel{\hat{z}}) + \cancel{\Delta\rho g \hat{z}}\right)$$

$$= -\frac{k \Delta\rho g}{\mu_f}\nabla h = -K \nabla h$$

$$\boxed{q_r = -K \nabla h}$$

$$K = \underbrace{\frac{k_0 \Delta\rho g}{\mu_f}}_{K_0} \phi^n = K_0 \phi^n$$

## Compaction relation

$$\nabla \cdot \underline{v}_s = \frac{p}{\xi} = \frac{\Delta\rho g}{\xi}(h-z) = \frac{h-z}{\Xi} \qquad \Xi = \frac{\xi}{\Delta\rho g}$$

capital Xi

## Continuity in terms of overpressure head

$$\boxed{-\nabla \cdot [K \nabla h] + \frac{h}{\Xi} = \frac{z}{\Xi} - \frac{\Delta\rho}{\rho_f \rho_s}\Gamma}$$

new part

just due to h-form

$\Rightarrow$ mod. Helmholtz equation

generates wave like behavior

## Porosity evolution

Solid mass balance

$$-\frac{\partial \phi}{\partial t} + \nabla \cdot [(1-\phi)\underline{v}_s] = -\frac{\Gamma}{\rho_s}$$

$$-\frac{\partial \phi}{\partial t} + \underbrace{\nabla \cdot \underline{v}_s}_{\frac{p}{\xi}} - \nabla \cdot [\underline{v}_s \phi] = -\frac{\Gamma}{\rho_s}$$

$$\boxed{\frac{\partial \phi}{\partial t} + \nabla \cdot \underbrace{[\underline{v}_s \phi]}_{\text{adv.}} = \frac{h-z}{\Xi} + \frac{\Gamma}{\rho_s}}$$

Three factors that affect porosity

1) porosity moves with solid velocity
2) over pressure $(p = h - z > 0)$ generates porosity
3) melting $(\Gamma > 0)$ generates porosity

Need to determine $\underline{v}_s$ !

<u>Solid velocity field</u>

Strictly we need to solve Stokes eqn for $\underline{v}_s$

For now we make an approximation

$$\underline{v}_s = \underbrace{-\nabla u}_{\text{dilation}} + \underbrace{\nabla \times \underline{\psi}}_{\text{shear.}}$$

Helmholtz decomposition

$u$ = scaler potential

$\underline{\psi}$ = vector potential

Assume $\nabla \times \underline{\psi}$ is negligible $\Rightarrow$ no shear

$$\boxed{\underline{v}_s = -\nabla u}$$

Substitute it into compaction relation

$$\nabla \cdot \underline{v}_s = \frac{p}{\xi} = \frac{h-z}{\Xi} \Rightarrow \boxed{-\nabla^2 u = \frac{h-z}{\Xi}}$$

Simplified model for melt migration in ductile ice comprise three coupled non-linear PDE's:

1) $$-\nabla \cdot [K(\phi) \nabla h] + \frac{h}{\Xi(\phi)} = \frac{z}{\Xi(\phi)} - \frac{\Delta\rho}{\rho_s \rho_f} \Gamma$$

2) $$-\nabla^2 u = \frac{h-z}{\Xi(\phi)}$$

Flow problem

3) $$\frac{\partial \phi}{\partial t} + \nabla \cdot [\underline{v}_s \phi] = \frac{h-z}{\Xi(\phi)} + \frac{\Gamma}{\rho_s}$$

Transport problem

with constitutive law:

$$\underline{v}_s = -\nabla u \quad \text{and} \quad q_r = -K \nabla h$$

$$K(\phi) = K_0 \phi^n \quad \text{and} \quad \Xi(\phi) = \frac{\Xi_0}{\phi^m}$$