Lecture 11: Melt migration - scaling & solutions

Logistics: - HW3 due

- sorry for office hours yesterday

Last time: Simplified melt migration equations

Compaction relation: $p = p_f - p_s = \xi \, \nabla \cdot \underline{v}_s = \dot{\varepsilon}_{vol}$

Darcy in terms of p: $q_r = -\frac{k}{\mu}(\nabla \underline{p} + \Delta \rho g \hat{z})$

Head formulation: $\underline{h} = \frac{p}{\Delta \rho g} + z$

$\Rightarrow \quad h - z = \Xi \, \nabla \cdot v_s$

$$\boxed{q = -K \nabla h}$$

Porosity dependent phys. properties:

$K = K_0 \, \phi^n \qquad \Xi = \frac{\Xi_0}{\phi^m}$

Governing equations:

1) $-\nabla \cdot [K(\phi) \nabla h] + \frac{h}{\Xi(\phi)} = \frac{z}{\Xi(\phi)} - \frac{\Delta \rho}{\rho_f \rho_s} \Gamma$

2) $-\nabla^2 u = \frac{h - z}{\Xi(\phi)}$

3) $\frac{\partial \phi}{\partial t} + \nabla \cdot [\underline{v}_s \phi] = \frac{h - z}{\Xi(\phi)} + \frac{\Gamma}{\rho_s}$

Today: Non-dimensionalize, Fundamental solns.

Scale the governing equations

Why?

1) Cleans up the equations?
2) Identify dimensionless gov. parameters!
3) Identify small terms that can be dropped!
4) Better scaling of equations

   ⇒ help the numerics

Governing equations:

1) $- \nabla \cdot [K_0 \phi^n \nabla h] + \frac{\phi^m}{\Xi_0} h - \frac{\phi^m}{\Xi_0} z - \frac{\Delta \rho}{\rho_f \rho_s} \Gamma$

2) $-\nabla^2 u = \frac{\phi^m}{\Xi_0} (h - z)$

3) $\frac{\partial \phi}{\partial t} + \nabla \cdot [\underline{v}_s \phi] = \frac{\phi^m}{\Xi_0}(h - z) + \frac{\Gamma}{\rho_s}$

on $x \in [0, L]$ $z \in [0, H]$ and $t \in [0, T]$

Scale all variables:

independent variables: $\underline{x}_D = \frac{\underline{x}}{x_c}$ $t_D = \frac{t}{t_D}$

primary dep. variables: $\phi_D = \frac{\phi}{\phi_c}$ $h_D = \frac{h}{h_c}$ $u_D = \frac{u}{u_c}$

secondary dep. variables: $\underline{v}_D = \frac{\underline{v}_s}{v_c}$ $\underline{q}_D = \frac{\underline{q}_r}{q_c}$ $\Gamma_D = \frac{\Gamma}{\Gamma_c}$

All variables with "D" subscript are dimensionless.
and characteristic variables $(x_c, t_c, \phi_c \ldots)$ are chosen to set magnitude of dim.-less variables to one.

What are these characteristic scales?
Some variables have external scales:

$x_c \rightarrow H, L$ $t_c \rightarrow T$

Typically we choose internal scales suggested by the equations themselves

Non-dimensionalize gov. eqns by substituting the scaled variables

$\phi = \phi_c \phi_D$ $t = t_c t_D$ $\underline{x} = x_c \underline{x}_D$ ....

for example: $\frac{\partial \phi}{\partial t} = \frac{\partial \phi_c \phi_D}{\partial t_c t_D} = \frac{\phi_c}{t_c} \frac{\partial \phi_D}{\partial t_D}$

$$\nabla\cdot = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial (x_c x_D)}, \frac{\partial}{\partial x_c y_D}, \frac{\partial}{\partial x_c z_D}\right)$$

$$= \frac{1}{x_c}\left(\frac{\partial}{\partial x_D}, \frac{\partial}{\partial y_D}, \frac{\partial}{\partial z_D}\right) = \frac{1}{x_c}\nabla_D\cdot$$

$$\nabla = \frac{1}{x_c}\nabla_D$$

I Our pressure equation

$$-\nabla\cdot[K_0\phi^n\nabla h] + \frac{\phi^m}{\Xi_0}h = \frac{\phi^m}{\Xi_0}z - \frac{\Delta\rho}{\rho_s\rho_f}\Gamma$$

substitute & collect terms   I have 'm' one that's wrong

(n) - in notes

$$-\frac{K_0\phi_c^n h_c}{x_c^2}\nabla_D\cdot[\phi_D^n\nabla_D h_D] + \frac{\phi_c^m h_c}{\Xi_0}\phi_D^m h_D = \frac{\phi_c^m x_c}{\Xi_0}z_D - \frac{\Delta\rho\Gamma_c}{\rho_f\rho_s}\Gamma_D$$

Introduce char. hydr. cond. & bulk viscosity

$$K_c = K_0\phi_c^n \qquad \Xi = \frac{\Xi_0}{\phi_c^m}$$

Choose to scale to div. term

$$-\nabla_D\cdot[\phi_D^n\nabla_D h_D] + \underbrace{\frac{x_c^2}{\Xi_c K_c}}_{\Pi_1}\phi_D^m h_D = \underbrace{\frac{x_c^3}{K_c\Xi_c h_c}}_{\Pi_2}\phi_D^m z_D - \underbrace{\frac{\Delta\rho\Gamma_c x_c^2}{K_c h_c\rho_f\rho_s}}_{\Pi_3}\Gamma_D$$

Three dimensionless groupings Π_1, Π_2 & Π_3

Assume IC gives a scale of $\phi \rightarrow \phi_c$

$\rightarrow K_c \quad \Xi_c$

$\Pi_1 = \dfrac{x_c^2}{K_c \Xi_c}$ this suggest an internal lengthscale

$\Pi_1 = 1 \quad \Rightarrow \quad x_c = \sqrt{K_c \Xi_c} = \sqrt{K_0 \phi_c^n \dfrac{\Xi_0}{\phi^m}} =$

$K_0 = \dfrac{k_0 \Delta\rho g}{\mu_f} \qquad \Xi_0 = \dfrac{\xi_0}{\Delta\rho g} \qquad \xi_0 = c\eta$

introduce $k_c = k_0 \phi_c^n \qquad \xi_c = \dfrac{\xi_0}{\phi_0^m}$

$$\boxed{x_c = \sqrt{\frac{k_c \xi_c}{\mu_f}}}$$ compaction length

Compaction length is important internal length scale in melt migration. The physical interpretation is the distance over which changes in overpressure/porosity can be communicated in the partially molten material.

Once x_c is known we look Π_c to get a scale for our pressure head

$$\Pi_2 = \frac{x_c^3}{\underbrace{K_c \Xi_c}_{x_c^2} h_c} = \frac{x_c^3}{x_c^2 h_c} = \frac{x_c}{h_c} = 1 \Rightarrow \boxed{h_c = x_c}$$

Finally we use Π_3 to suggest Γ_c

$$\Pi_3 = \frac{\Delta\rho\, \Gamma_c\, x_c^2}{K_c h_c \rho_f \rho_s} = 1 \Rightarrow \boxed{\Gamma_c = \frac{K_c \rho_f \rho_s}{x_c \Delta\rho}}$$

Note this simply sets coefficient in melting term to 1. If we had a proper melting model it would suggest its own Γ_c

With these choices we have

$$\boxed{-\nabla_D \cdot [\phi_D^n \nabla_D h_D] + \phi_D^m h_D = \phi_D^m z_D - \Gamma_D}$$

Darcy: $$q = -K_0 \phi^{\overset{n}{\cancel{m}}} \nabla h$$

$$q_c q_D = -\underbrace{K_0 \phi_c^{\overset{n}{\cancel{m}}}}_{K_c} \phi_D^{\overset{n}{\cancel{m}}} \overset{1}{\cancel{\frac{h_c}{x_c}}} \nabla_D h_D$$

$$\boxed{q_c = K_c} \quad \Rightarrow \quad \boxed{q_D = -\phi_D^n \nabla_D h_D}$$

II) Equation for velocity potential

$$-\nabla^2 u = \frac{\phi^m}{\Xi_0}(h-z)$$

$$-\frac{u_c}{x_c^2} \nabla_D^2 u_D = \underbrace{\frac{\phi_c^m}{\Xi_0}}_{\Xi_c} \phi_D^m \, x_c (h_D - z_D)$$

$$-\nabla_D^2 u_D = \underbrace{\frac{x_c^3}{\Xi_c u_c}}_{\Pi_4} \phi_D^m (h_D - z_D)$$

set $\Pi_4 = \frac{x_c^3}{\Xi_c u_c} = 1$ $\qquad u_c = \frac{x_c^3}{\Xi_c} = \frac{K_c \Xi_c x_c}{\Xi_c} = K_c x_c$

$$\boxed{u_c = K_c x_c}$$

dim. less. $\boxed{-\nabla_D^2 u_D = \phi_D^m (h_D - z_D)}$

This also implies a scale for solid velocity

$\boxed{\underline{v}_s = -\nabla u}$ $\qquad \underline{v}_D v_c = -\frac{u_c}{x_c} \nabla_D u_D$

$$\underline{v}_D = -\underbrace{\frac{u_c}{x_c v_c}}_{=1} \nabla_D u_D \quad \Rightarrow \quad \boxed{v_c = \frac{u_c}{x_c} = K_c}$$

III Scale porosity evolution equation

$$\frac{\partial \phi}{\partial t} + \nabla \cdot [v_s \phi] = \frac{\phi^m}{\Xi_0}(h-z) + \frac{\Gamma}{\rho_s}$$

$$\frac{\phi_c}{t_c}\frac{\partial \phi_D}{\partial t_D} + \frac{v_c \phi_c}{x_c}\nabla_D \cdot [v_D \phi_D] = \underbrace{\frac{\phi_c^m}{\Xi_0}}_{\Xi_c} x_c\, \phi_D^m (h_D - z_D) + \frac{\Gamma_c}{\rho_s}\Gamma_D$$

scale to accumulation term

$$\frac{\partial \phi_D}{\partial t_D} + \underbrace{\frac{v_c t_c}{x_c}}_{\Pi_5} \nabla_D \cdot [v_D \phi_D] = \underbrace{\frac{x_c t_c}{\Xi_c \phi_c}}_{\Pi_6} \phi_D^m (h_z - z_D) + \underbrace{\frac{\Gamma_c t_c}{\rho_s \phi_c}}_{\Pi_7} \Gamma_D$$

Three dim. less groupings that suggest internal time scales:

1) Advective: $\Pi_5 = \frac{v_c t_c}{x_c} = 1 \Rightarrow t_c = \frac{x_c}{v_c} = \frac{x}{v_s}$

2) Compaction: $\Pi_6 = \frac{x_c t_c}{\Xi_c \phi_c} = 1 \Rightarrow t_c = \frac{\phi_c \Xi_c}{x_c}$

3) Reactive: $\Pi_7 = \frac{\Gamma_c t_c}{\rho_s \phi_c} = 1 \Rightarrow t_c = \frac{\rho_s \phi_c}{\Gamma_c}$

Here we choose compaction time scale

$$\frac{\partial \phi_D}{\partial t} + \underbrace{\frac{v_c \phi_c \Xi_c}{x_c^2}}_{\phi_c} \nabla_D \cdot (v_D \phi_D) = \phi^m (h_D - z_D) + \underbrace{\frac{\Gamma \Xi}{\rho_s x_c}}_{Da = \frac{\rho_f}{\Delta \rho}} \Gamma_D$$

Full dimless system

$$\boxed{\begin{aligned}
&1)\quad -\nabla_D \cdot [\phi_D^n \nabla_D h_D] + \phi_D^m h_D = \phi_D^m z_D - \Gamma_D \\
&2)\quad -\nabla_D^2 u_D = \phi_D^m (h_D - z_D) \\
&3)\quad \frac{\partial \phi_D}{\partial t_D} + \phi_c \nabla_D \cdot [v_D \phi_D] = \phi_D^m (h_D - z_D) + Da\, \Gamma_D
\end{aligned}}$$

$x_D \in [0, \frac{L}{x_c}]$ $\quad z_D \in [0, \frac{H}{x_c}]$ $\quad t_D \in [0, \frac{T}{t_c}]$

$\underline{q}_D = -\phi_D^m \nabla_D h_D$ $\quad \underline{v}_D = -\nabla_D u_D$