Lecture 12: Solving the flow problem

Logistics: - HW3 due date shifted to Tue (3/1/22)

- HW4 will be posted due Th (3/3/22)

Last time: - scaled governing equs

$$1)\quad -\nabla_D\cdot[\phi_D^n\nabla_D h_D] + \phi_D^m h_D = \phi_D^m z = -\Gamma_D$$

$$2)\quad -\nabla_D^2 u_D = \phi_D^m(h_D - z_D)$$

Flow instant.

$$3)\quad \frac{\partial\phi_D}{\partial t_D} + \phi\,\nabla_D\cdot[\underline{v}_D\phi_D] = \phi^m(h_D - z_D) + Da\,\Gamma_D$$

on $z_D \in [0, \frac{H}{x_c}]$

- Compaction length: $x_c = \frac{\delta}{H} = \delta = \sqrt{K_c \Xi_c} = \sqrt{\frac{k_c \xi_c}{\mu_f}}$
- Dim.less governing param.: $\frac{H}{\delta}$, $\cancel{\frac{\delta}{H}}$, ϕ_c, Da

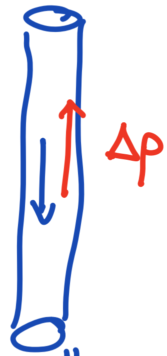
Today: - Study "Flow Problem" (Equs 1 & 2)

- Fundamental analytic solutions
- Numerical solution

Steady exchange flow

Assume no melting $\Gamma_D = 0$

Exchange flow?



ice & melt flow in

opposite directions due

to get Δp "in the same place".

Steady solution: $\phi = \text{const} = \phi_c \Rightarrow \phi_D = 1$

not write "D" subscripts

$$\cancelto{0}{\frac{\partial \phi}{\partial t}} + \phi_c \nabla \cdot [\underline{v}\,\cancelto{1}{\phi}] = \cancelto{1}{\phi^m}(h-z)$$

$$\phi_c \nabla \cdot \underline{v} = h - z = p$$

substitute compaction $\cancelto{1}{\phi^m}(\underbrace{h-z}_{p}) = \nabla \cdot v$

$$\Rightarrow \quad \phi_c\, p \overset{\Downarrow}{=} p \qquad p = 0$$

$\Rightarrow$ zero over pressure

$\Rightarrow$ Equ3 is trivially satisfied

$\phi^{-m} z$

Equ 1, $-\nabla\cdot \cancel{\phi^{\mu}}\nabla h + \cancel{\phi^{\mu}}\underbrace{\cancel{(h-z)}}_{p} = 0$

$$-\nabla^2 h = 0$$

Equ 2: $-\nabla^2 u = \cancelto{1}{\phi^{\mu}}\cancelto{0}{(h-z)}$

$$-\nabla^2 u = 0$$

The problem reduces to

$$\phi = 1 \qquad -\nabla^2 h = 0 \qquad -\nabla^2 u = 0$$

Integrating twice: $\left.\begin{aligned} h &= a_1 z + a_2 \\ u &= b_1 z + b_2 \end{aligned}\right\}$ linear

Typically coeff. are determined from BC but here we imagine an infinite column

$\Rightarrow$ can't apply BC's

From 3 we know $p = h - z = 0$

$$a_1 z + a_2 - z = 0 \quad \Rightarrow \quad a_1 = 1 \qquad a_2 = 0$$

$$\Rightarrow \quad \boxed{h = z}$$

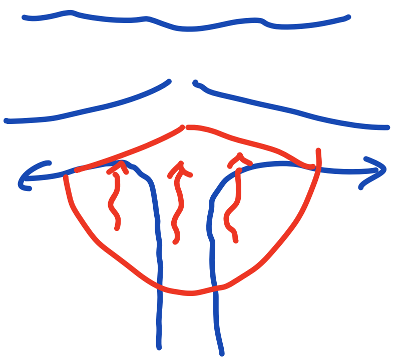
Darcy's law: $q = -\cancelto{1}{\phi^{\mu}}\dfrac{dh}{dz} = -1 \qquad \boxed{q = -1}$

From continuity: $\nabla \cdot [\underline{q}_r + \underline{v}_s] = 0$ (dimensional)

$q_c = v_c = K_c$ $\qquad \frac{d}{dz}[q + v] = 0$ (dim-less)

$$q + v = \text{const.}$$

This const. is also determined by BC and sets the "net" motion of the exchange flow.



Example: during melt extraction beneath mid-ocean riges the net flow is not zero.

But in our case brine drainage is entirely compensated by upward ice compaction

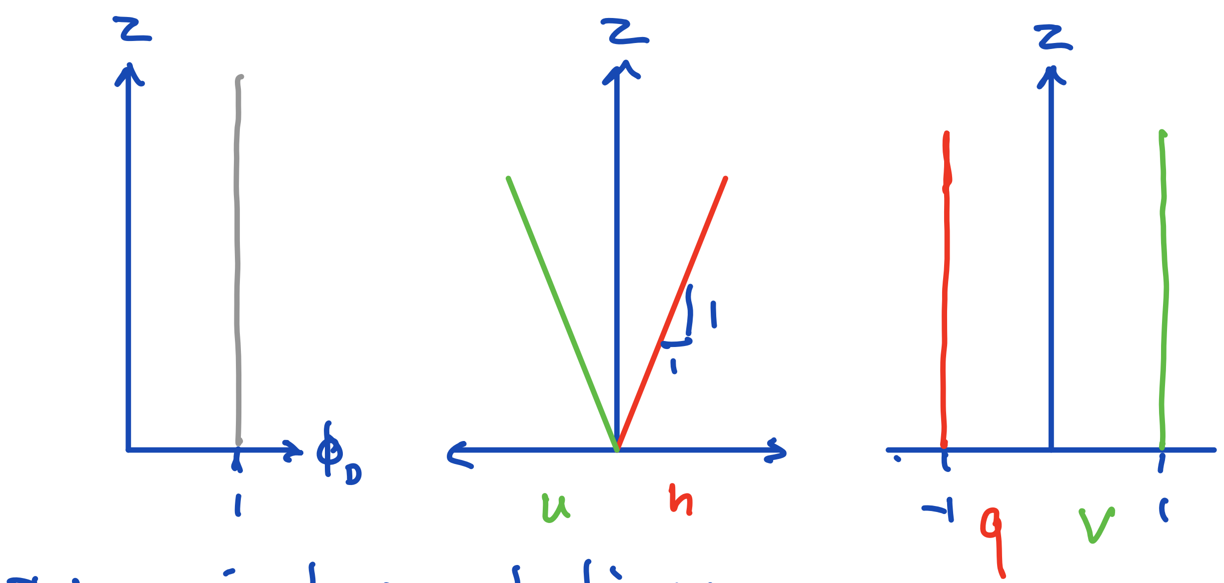
$\Rightarrow$ zero net motion $\Rightarrow$ const = 0

$$\Rightarrow \quad \boxed{v = -q = 1}$$

Velocity potential is $\quad v = -\frac{du}{dz}$

$\frac{du}{dz} = 1 \qquad u = b_1 z + b_2 \qquad b_1 = -1$

$\Rightarrow \quad \boxed{u = -z + b_2}$ where b_2 is arbitrary



Dimensionless solution:

$\phi_D = 1$, $h_D = z_D$, $u_D = -z_D$, $q_D = -1$ & $v_D = 1$

This does not mean that ice & melt move with same velocity but opposite directions.

Redimensionalize

$q_r = q_c\, q_D = -K_c$ $\qquad v_s = v_c\, v_D = K_c$ $\qquad \phi = \phi_c\, \phi_D = \phi_c$

def. of relative flux: $q_r = \phi\,(v_f - v_s)$

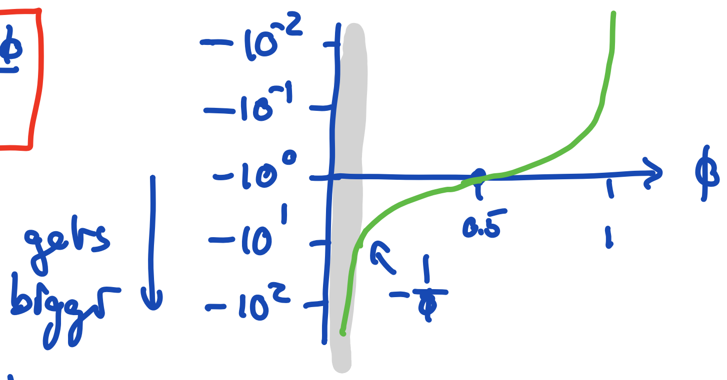
substitute: $-K_c = \phi\,(v_f - K_c)$

$$-\frac{K_c}{\phi} = v_f - K_c$$

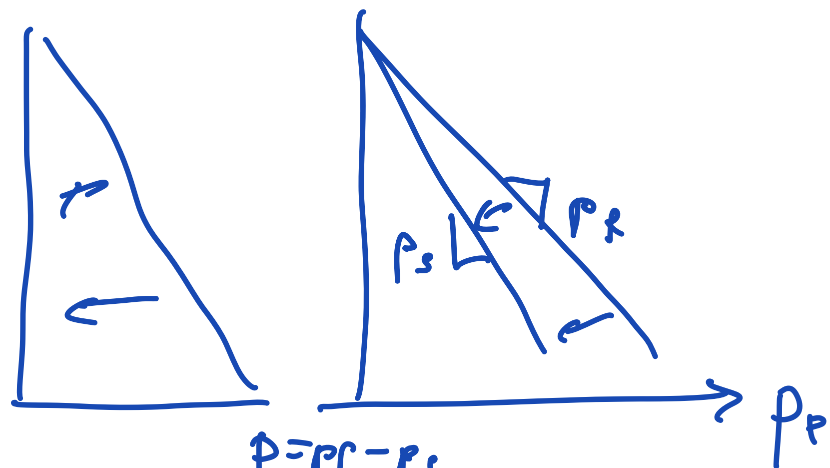
$$v_f = K_c - \frac{K_c}{\phi} = \underset{\substack{\uparrow \\ v_s}}{K_c}\left(1 - \frac{1}{\phi}\right)$$

$$\frac{v_f}{v_s} = 1 - \frac{1}{\phi} = \frac{\phi - 1}{\phi} = -\frac{1-\phi}{\phi} = \frac{\phi_s}{\phi_f}$$

$$\boxed{\frac{v_f}{v_s} = -\frac{1-\phi}{\phi}}$$



$$\lim_{\phi \ll 1} \frac{v_f}{v_s} = -\frac{1}{\phi}$$



$p = p_f - p_s$

The flow is exactly so fast as to reduce the fluid pressure gradient to lithostatic.

$\Rightarrow$

Finite compacting column

height is H dimensionless height $\frac{H}{\delta}$

closed ~~boun~~ top & bottom boundaries

$\underline{v}_s = q_r (= v_f) = 0$ at $z_D = 0$ $z_D = \frac{H}{\delta}$

constant porosity $\phi = \phi_c$ $\phi_D = 1$

Instantaneous flow problem (u, h, q, v)

1) $-\frac{d^2h}{dz^2} + h = z$ with $q = -\frac{dh}{dz}\Big|_0 = -\frac{dh}{dz}\Big|_{\frac{H}{\delta}} = 0$

2) $-\frac{d^2u}{dz^2} = h - z$ with $v = -\frac{du}{dz}\Big|_0 = -\frac{du}{dz}\Big|_{\frac{H}{\delta}} = 0$

Eqn non-homogeneous (has rhs) 2^{nd} order ODE with const coefficients.

$\Rightarrow$ solve with method of undetermined coeff.

$h = h_h + h_p$

guess $h_p = c_3 z$

$h_h = c_1 e^{r_1 z} + c_2 e^{r_2 z}$

subst. e^{rz} into $-\frac{d^2h}{dz^2} + h = 0$

$$-r^2 e^{rz} + e^{rz} = 0$$

$$-r^2 + 1 = 0 \qquad r = \pm 1$$

$$\Rightarrow \quad h_h = c_1 e^{z} + c_2 e^{-z}$$

subst h_p into $-\frac{d^2h}{dz^2} + h = z$

$$h_p = c_3 z \qquad\qquad 0 + c_3 z = z \quad \Rightarrow c_3 = 1$$

General soln: $h = c_1 e^{z} + c_2 e^{-z} + z$

determine c_1 and c_2 from BC.

$$\cdot \quad \frac{dh}{dz} = c_1 e^{z} - c_2 e^{-z} + 1$$

BC: $\left.\frac{dh}{dz}\right|_0 = c_1 - c_2 + 1 = 0 \qquad c_2 = c_1 + 1$

$$\left.\frac{dh}{dz}\right|_{\frac{H}{\delta}=H_D} = c_1 e^{H_D} - c_2 e^{-H_D} + 1 = 0$$

solve for c_1 & c_2: $c_1 = \frac{e^{-H_D}-1}{e^{H_D}-e^{-H_D}} \qquad c_2 = \frac{e^{H_D}-1}{e^{H_D}-e^{-H_D}}$

Solution is:

$$\boxed{\begin{aligned} h(z) &= z + \frac{e^{-H_D}-1}{e^{H_D}-e^{-H_D}}\, e^{z} + \frac{e^{H_D}-1}{e^{H_D}-e^{-H_D}}\, e^{-z} \\ q(z) &= -1 - \frac{e^{-H_D}-1}{e^{H_D}-e^{-H_D}} + \frac{e^{H_D}-1}{e^{H_D}-e^{-H_D}}\, e^{-z} \end{aligned}}$$

exchange flow boundary layers
at top & bottom