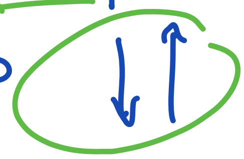
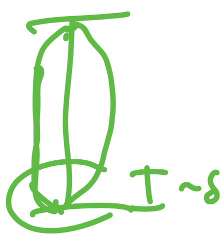
Lecture 13: Advection Equation

Logistics: - HW 4 due Th

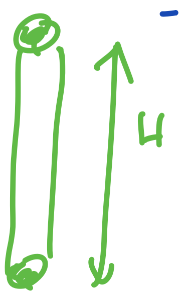
Last time: - Analytic solutions to flow problem

- Steady exchange flow 

$\phi_D = 1,\ h_D = z_D,\ u_D = -z_D,\ q_D = -1,\ v_D = 1$ $\quad P_D = 0$



The pressure gradient due to flow exactly balances the difference between hydrostatic & lithostatic pressure



- Compacting column
  - → form boundary layers $\sim \delta$
  - if $H \gg \delta$: recover steady exchange flow in center.
- Numerical solution

Today: - Transport problem: Porosity evolution

- Advection eqn
- Method of Characteristics & Upwinding

Porosity Evolution Equation (Transport problem)

$$\frac{\partial \phi_D}{\partial t_D} + \phi_c \underbrace{\nabla_D \cdot [\underline{v}_D \phi_D]}_{\text{Advective flux}} = \underbrace{\phi_D^m (h_D - z_D) + \mathcal{D}a \Gamma_D}_{\text{source terms}}$$

$\Rightarrow$ focus on advective flux.

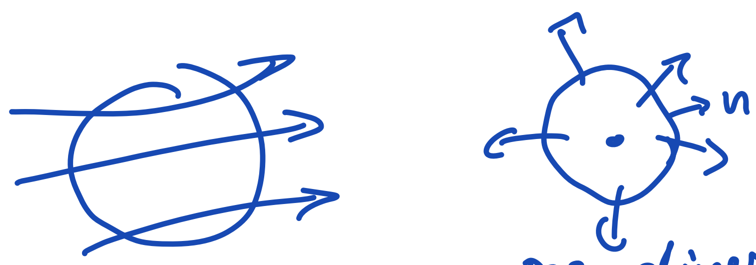
$\Rightarrow$ rhs $= 0$ this applies in center of ~~a~~ flow
exchange

if $p_D = h_D - z_D = 0$ compaction $\nabla_D \cdot \underline{v}_D = 0$

$$p_D = \phi_D^m \nabla \cdot v_D$$

Rewrite the advective flux:

$$\nabla_D \cdot [\underline{v}_D \phi_D] = \underline{v}_D \cdot \nabla \phi_D + \phi_D \cancelto{0}{\nabla_D \cdot \underline{v}_D} = \underline{v}_D \cdot \nabla \phi_D$$



Divergence free   pos. divergence

$$\int_\Omega \nabla \cdot v \, ds = \oint_{\partial\Omega} \underline{v} \cdot n \, ds$$

More common form of advection eqn:

$$\frac{\partial \phi_D}{\partial t_D} + \phi_c \underline{v}_D \cdot \nabla \phi_D = 0$$

In 1 dimension $\phi_c \underline{v} = v$

$$\boxed{\frac{\partial \phi_D}{\partial t_D} + v \frac{\partial \phi_D}{\partial x_D} = 0}$$

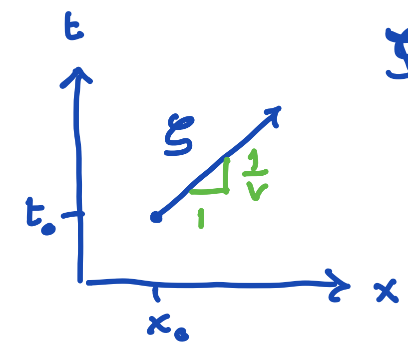
Method of characteristics

Drop 'D' subscripts

PDE : $\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial x} = 0 \qquad x \in \mathbb{R}$

IC : $\phi(x, t=0) = \phi_0(x)$

Idea: Find a characteristic curve/coordinate ξ along which the PDE reduces to an ODE.



$$\phi(x,t) = \phi(x(\xi), t(\xi)) = \Phi(\xi)$$

Total change of Φ along ξ is

$$\frac{d\Phi}{d\xi} = \frac{\partial \phi}{\partial t}\frac{dt}{d\xi} + \frac{\partial \phi}{\partial x}\frac{dx}{d\xi} = 0 \quad \text{compare with PDE}$$

PDE : $\frac{\partial \phi}{\partial t} \cdot 1 + \frac{\partial \phi}{\partial x} v = 0$

By comparison:

$\frac{d\Phi}{d\xi} = 0 \Rightarrow$ solution does not change along the characteristic curve.

$\frac{dt}{dx} = \frac{1}{v}$

$$\boxed{\frac{dt}{d\xi} = 1 \qquad \frac{dx}{d\xi} = v} \Rightarrow \frac{dx}{dt} = v \qquad \text{" char. equ "}$$

solve: $x - x_0 = v\,(t - t_0)$

from IC: $\phi(x = x_0, t = t_0) = \phi_0(x_0)$

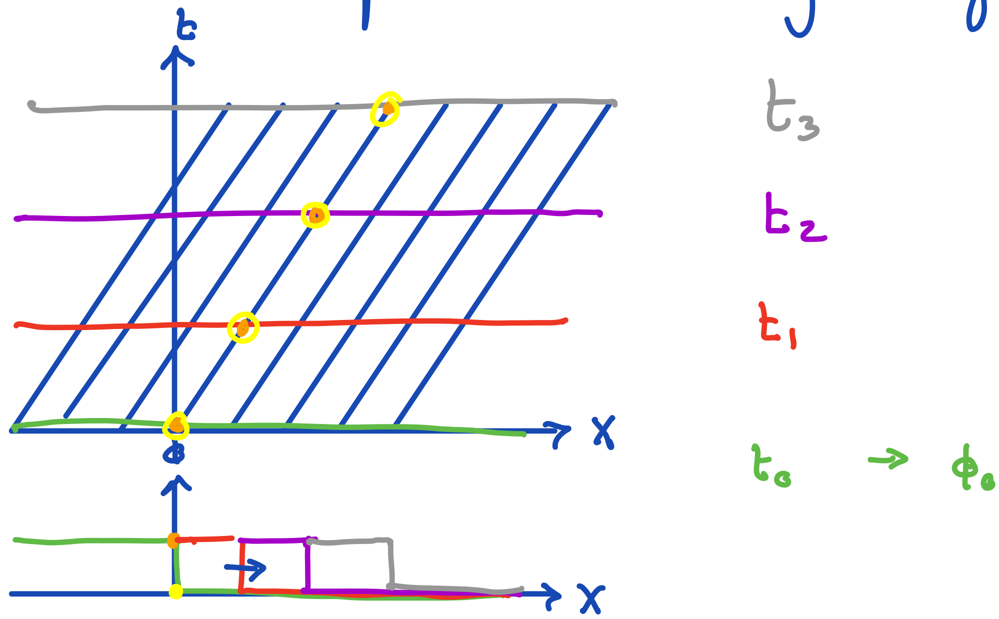
solve char. equ: $x_0 = x - v\,(t - t_0)$

subst into IC: $\boxed{\phi(x,t) = \phi_0(x - v(t - t_0))}$

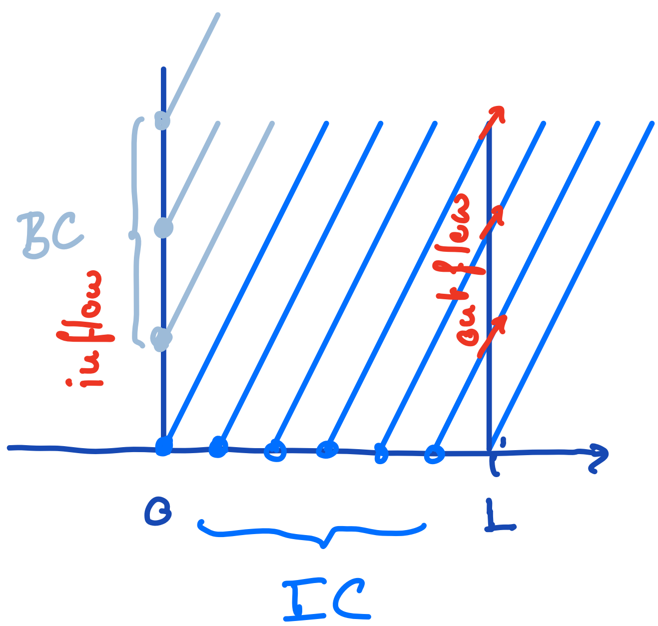
general solution to advection equation for any ϕ_0

$x - vt \equiv$ travelling wave coordinate. $(t_0 = 0)$

Initial profile, ϕ_0, simply translates with constant shape and velocity to right $(v > 0)$



Consider a finite domain

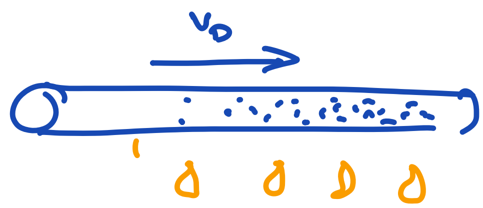


- don't need BC on outflow side
- need BC on inflow side
- in multi-phase flow in and out flow depends on phase

Steady advection with melting

DDE: $\phi_c \nabla\cdot[v_D \phi_D] = Da\, \Gamma_D$ $\qquad x_D \in [0,1]$

BC: $\phi_D(x_D = 0) = 0$



"Pushing column of ice over the fire"

In 1D $\quad v_D = 1 \quad \Gamma_D = 1 \quad \phi_c = Da$

$\Rightarrow \quad \frac{\partial \phi_D}{\partial x_D} = 1$ integrate $\boxed{\phi_D = x_D}$

Discretization of steady advection

Continuous: $\nabla_D \cdot [\underline{v}_D \phi_D] = 1$

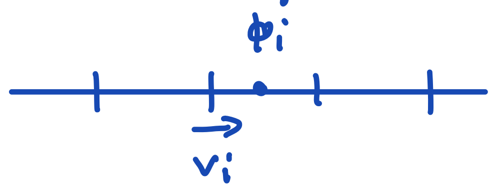
$q_D = \underline{v}_D \phi_D$ advective flux

Discretely: $\underline{\underline{D}}\, \underline{q} = \underline{f}_s$

$\underline{q}$ = discrete adv. flux vector

$(\underline{q} = -\underline{Kd}\, \underline{\underline{G}}\, \underline{h})$

How to compute $\underline{q}$ from $\underline{v}$ and $\underline{\phi}$?



$\underline{v}$ = Nf by 1 on faces

$\underline{\phi}$ = N by 1 ~~on~~ in cells

Advection matrix: $\underline{q} = \underline{\underline{A}}\ \underline{\phi}$

$Nf \cdot 1 \quad Nf \cdot N \quad N \cdot 1$

$\underline{\underline{A}}$ is Nf by N matrix that computes

$\underline{q}$ from ϕ and $\underline{v}$

$\Rightarrow \underline{\underline{A}} = \underline{\underline{A}}(\underline{v})$

shape of $\underline{\underline{A}}$ is same as $\underline{\underline{G}}$

In this case we discretize advection equ:

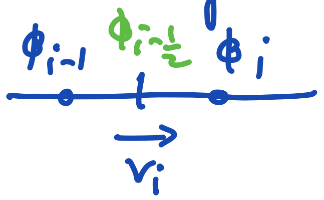
$$\nabla_D \cdot [\underline{v}_D \, \phi_D] = 1$$

$$\underbrace{\underline{\underline{D}}\,[\underline{\underline{A}}(\underline{v})]}_{\underline{\underline{L}}(\underline{v})}\,\underline{\phi} = \underline{f}_s$$

$$\underline{\underline{L}}(\underline{v})\,\underline{\phi} = \underline{f}_s$$

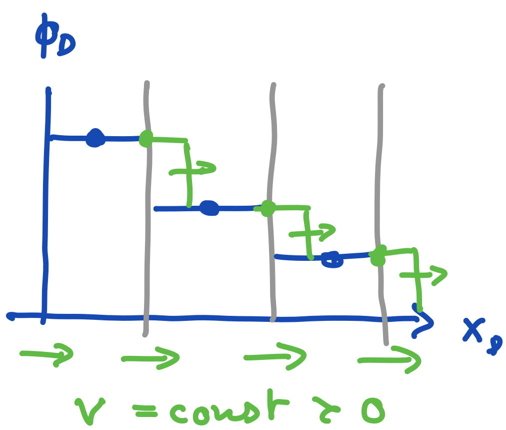
Construction of $\underline{\underline{A}}$

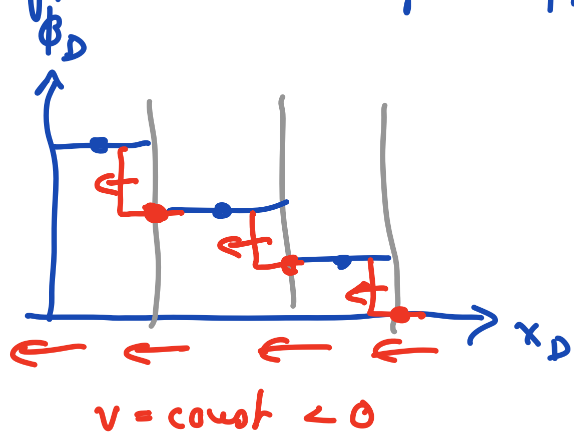
The purpose of $\underline{\underline{A}}$ is to estimate ϕ on the cell faces and multiply by $\underline{v}$



$$a_i = v_i \, \phi_{i-1/2}$$

Use our understanding from MOC to find $\phi_{i-1/2}$



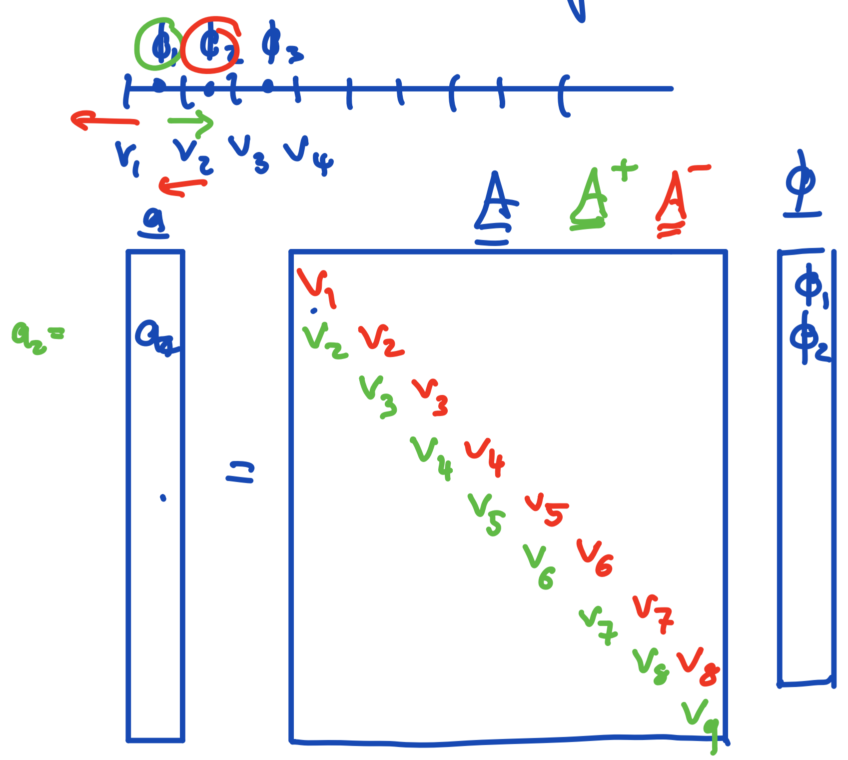


$v_i > 0 : \quad \phi_{i-1/2} = \phi_{i-1}$ $\qquad$ $v_i < 0 : \quad \phi_{i-1/2} = \phi_i$

From analytic soln we know that ϕ only depends on "upstream/upwind" value of ϕ

Natural choice: $\boxed{\phi_{i-1/2} = \begin{cases} \phi_{i-1} & v > 0 \\ \phi_i & v < 0 \end{cases}}$

Construction of A



$a_2 = v_2 \phi_1$

$a_3 = v_3 \phi_2$

$a_2 = v_2 \phi_2$

A is sparse diagonal matrix

negative velocities go on main diagonal

positive velocities on -1 diagonal

Build $\underline{vp}$ and $\underline{vn}$ vectors as follows

$$\underline{vn} = \min(\underline{v}(1:Nx), 0)$$

$$\underline{vp} = \max(\underline{v}(2:Nx+1), 0)$$

$$\underline{v} = \begin{bmatrix} -1 \\ 7 \\ 4 \\ -3 \\ -2 \end{bmatrix} \qquad \underline{vp} = \begin{bmatrix} 0 \\ 7 \\ 4 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{vn} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -3 \\ 2 \end{bmatrix}$$