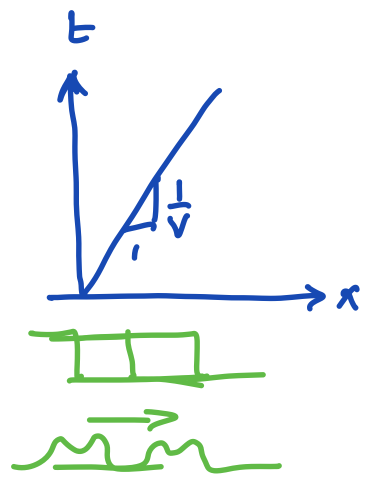
Lecture 14: Time stepping

Logistics: - HW4 is due

- HW5 will be posted
- Next week (3/8 and 3/10) zoom lectures

Last time: - Advection equation

$$\frac{\partial \phi_D}{\partial t_D} + \phi_c \nabla_D \cdot [\underline{v}_D \phi_D] = 0$$



- Method of characterisics

$$\boxed{\phi(x,t) = \phi_0(x - vt)}$$

travelling wave solution

- BC only on inflow side
- Steady melting of translating ice

$$\phi_D = x_D$$

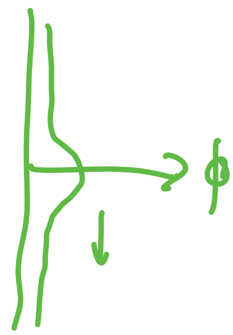
- Discretization of steady adv. equ

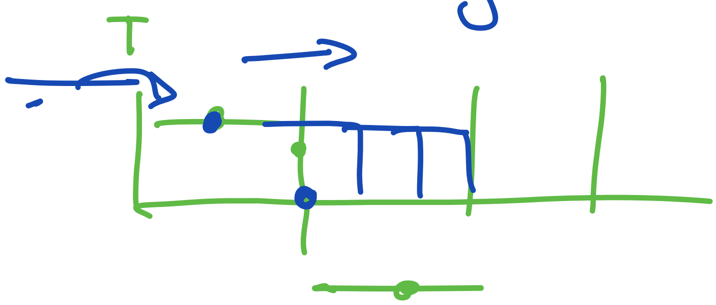
$$\nabla_D \cdot (\underline{v}_D \phi_D) \approx \underline{\underline{D}}\, \boxed{\underline{\underline{A}}(\underline{v})}\, \underline{\phi} \qquad q_i = v_i \phi_{i-1/2}$$

- Upwinding $\quad \phi_{i-1/2} = \begin{cases} \phi_{i-1} & v_i \geq 0 \\ \phi_i & v_i < 0 \end{cases}$

Today: - Time stepping & porosity evolution

- Theta method
- Time step restriction
- Transient compaction
- Solitary wave solution





$$\frac{\partial T}{\partial t} + v\cdot\nabla T = 0$$

Solving porosity evolution equation

PDE

$p_D \quad \phi_D$

$$\frac{\partial \phi_D}{\partial t} + \phi_c \nabla_D\cdot[\underline{v}_D \phi_D] = \phi_D^{m}(\underbrace{h_D - z_D}_{p_D}) + \Gamma_D$$

From solving flow problem $\rightarrow h_D \rightarrow p_D$

$\underline{v}_D$

Only unknown is ϕ_D but equ is non-linear

in ϕ_D ~~etc~~ if $m \neq 0, 1$

For ~~wo~~ now we assume $m=1 \Rightarrow$ linear

Semi-discrete form (drop 'D')

$$\frac{\partial \phi}{\partial t} + \phi_c \underline{\underline{D}}\, \underline{\underline{A}}(\underline{v})\, \underline{\phi} = \underline{\underline{Pd}}\, \underline{\phi} + \underline{\Gamma}$$

$$\frac{\partial \phi}{\partial t} + \underbrace{\phi_c \left[\underline{\underline{D}}\, \underline{\underline{A}}(\underline{v}) - \underline{\underline{Pd}}\right]}_{\underline{\underline{L}}} \underline{\phi} = \underline{\Gamma}$$

$$\underline{\underline{Pd}} = \begin{bmatrix} \ddots & & \\ & P_D & \\ & & \ddots \end{bmatrix}$$

$N_x \cdot N_x$

$$\frac{\partial \phi}{\partial t} + \underline{\underline{L}}\, \underline{\phi} = \underline{f_s} \qquad \underline{f_s} = \underline{\Gamma}$$

The only new part is time derivative

Theta-method

$$\frac{\partial \phi}{\partial t} \approx \frac{\underline{\phi}^{n-1} - \underline{\phi}^{n}}{\Delta t} = \underline{\underline{I}}\, \frac{\underline{\phi}^{n+1} - \underline{\phi}^{n}}{\Delta t}$$

$$\underline{\underline{I}}\,(\underline{\phi}^{n+1} - \underline{\phi}^{n}) + \Delta t\, \underline{\underline{L}}\, \underline{\phi}^{?} = \Delta t\, \underline{f_s}$$

choose time level for $\underline{\phi}$ in $\underline{\underline{L}}$ term

Theta method: $\underline{\phi} = \theta\, \underline{\phi}^{n} + (1-\theta)\, \underline{\phi}^{n+1}$

$$I(\underline{\phi}^{n+1} - \underline{\phi}^{n}) + \Delta t\, \underline{\underline{L}}\left(\theta\, \underline{\phi}^{n} + (1-\theta)\, \underline{\phi}^{n+1}\right) = \Delta t\, f_s$$

$\underline{\phi}^{n}$ is known $\rightarrow$ collect on r.h.s

$$\underbrace{\left(\underline{\underline{I}} + \Delta t \,(1-\theta)\, \underline{\underline{L}}\right)}_{\underline{\underline{Im}}} \underline{\phi}^{n+1} = \underbrace{\left(\underline{\underline{I}} - \Delta t \,\theta\, \underline{\underline{L}}\right)}_{\underline{\underline{Ex}}} \underline{\phi}^{n} + \Delta t \, f_s$$

At every time step we have to solve:

$$\boxed{\underline{\underline{Im}}\, \underline{\phi}^{n+1} = \underline{\underline{Ex}}\, \underline{\phi}^{n} + \Delta t \, f_s}$$

Linear system for time step

Properties of Theta-method

For $\theta = 1$: Forward Euler Method

$$\underline{\underline{Im}} = \underline{\underline{I}} \quad \Rightarrow \quad \underline{\phi}^{n+1} = \underline{\underline{Ex}}\, \underline{\phi}^{n} + \Delta t \, f_s$$

$\Rightarrow$ explicit method

only requires matrix-vector multiplication

$\Rightarrow$ cheap (per time step)

- first-order accurate

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x}$$

- conditionally stable

$\Rightarrow$ limit on timestep: $\Delta t \leq \frac{\Delta x}{v}$ general

for our specific case $\Delta t \leq \frac{\Delta x}{\phi_c \max(\underline{v})}$

For $\theta = 0$: Backward Euler

$\underline{\underline{Ex}} = \underline{\underline{I}}$

$\underline{\underline{Im}}$ is <u>not</u> diagonal

$\Rightarrow$ need to solve linear system $\rightarrow$ implicit

more expensive per timestep than explicit

- first-order in time
- unconditionally stable

For $\theta = \frac{1}{2}$: Crank-Nicholson / Trapezoidal rule

Neither $\underline{\underline{Im}}$ nor $\underline{\underline{Ex}}$ are $\underline{\underline{I}}$

$\Rightarrow$ need to solve system $\rightarrow$ implicit

- second order accurate
- unconditionally stable

  but has an oscillation limit

  (at least for diffusion problems)