# Lecture 15: 2D discrete operators

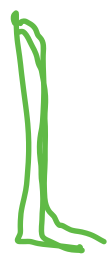
Logistics: HW5 has been posted

Last time: - Transport problem

- Time stepping:

$$\underline{\underline{I}}\,\frac{\underline{\phi}^{n+1}-\underline{\phi}^{n}}{\Delta t} + \underline{\underline{L}}\,\underline{\phi}^{?} = \underline{f}_s$$

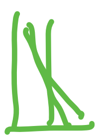
- Theta method: $\underline{\underline{L}}(\theta\underline{\phi}^{n} + (1-\theta)\underline{\phi}^{n+1})$



$\theta = 1$: Forward Euler, explicit, $1^{st}$-order

$\theta = 0$: Backward Euler, implicit, $1^{st}$-order

$\theta = \frac{1}{2}$: Crank-Nicholson, implicit, $2^{nd}$ order



- $$\boxed{\underline{\underline{Im}}\,\underline{\phi}^{n+1} = \underline{\underline{Ex}}\,\underline{\phi}^{n} + \underline{f}_s}$$

$\Rightarrow$ use solve_lbvp.m

- Transient compacting columns

Today: - Discretization in 2D

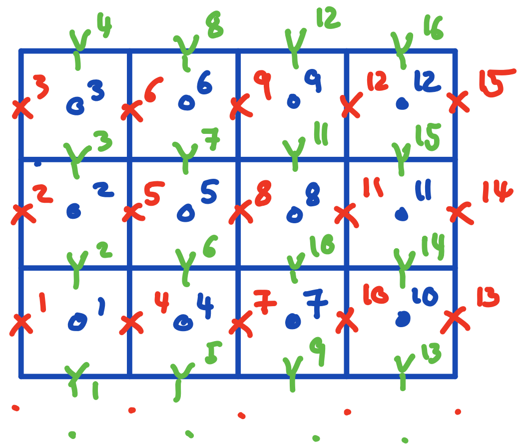
- Matlab basics

- Discrete operators in 2D

$\Rightarrow$ Tensor / Kronecker Products

# Discrete operators in 2D

Staggered grid in 2D



$N_x = 4 \quad N_y = 3 \quad N = 12$

x-faces: $N_{fx} = N_y(N_x+1) = 15$

y-faces: $N_{fy} = (N_y+1)\,N_x = 16$

Total faces: $N_f = N_{fx} + N_{fy} = 31$

## Discrete Gradient

Continuous gradient $\nabla h = \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{pmatrix}$

approx. $\frac{\partial h}{\partial x} \sim \underline{dhx}$ on x-faces

approx $\frac{\partial h}{\partial y} \sim \underline{dhy}$ on y-faces

Choose to build $\underline{\underline{G}}$ such that the resulting gradient vector is ordered as

$$\underline{dh} = \begin{bmatrix} \underline{dhx} \\ \underline{dhy} \end{bmatrix}$$

$\Rightarrow$ 2D gradient matrix can be decomposed as

$\underline{dh} = \underline{\underline{G}}\,\underline{h} \quad \Rightarrow \quad \underline{\underline{G}} = \begin{bmatrix} \underline{\underline{Gx}} \\ \underline{\underline{Gy}} \end{bmatrix}$ so that $\underline{dhx} = \underline{\underline{Gx}}\,\underline{h}$

$\underline{dhy} = \underline{\underline{Gy}}\,\underline{h}$

## Shape of gradient matrices

$$
\begin{array}{cccc}
 & \underline{dh} & \underline{\underline{G}} & \underline{h} \\
N_f\left\{\begin{array}{l} N_{fx} \\ \\ N_{fy} \end{array}\right. &
\left[\begin{array}{c} \underline{dhx} \\ \hline \underline{dhy} \end{array}\right] = &
\left[\begin{array}{c} \underline{\underline{Gx}} \\ \hline \underline{\underline{Gy}} \end{array}\right] &
\underbrace{\left[\begin{array}{c} \\ \\ \end{array}\right]}_{1}\Big\} N
\end{array}
$$

Matrix dimensions

$\underline{\underline{G}}$ $N_f \times N$

$\underline{\underline{Gx}}$ $N_{fx} \cdot N$

$\underline{\underline{Gy}}$ $N_{fy} \cdot N$

## Discrete divergence matrix

$$f_s = \nabla \cdot \overset{q = -dh}{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \approx \underline{\underline{D}}\, q = \underline{\underline{Dx}}\, \underline{qx} + \underline{\underline{Dy}}\, \underline{qy}$$

$$q = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$$

$$N\left\{\underbrace{\underline{f_s}}_{1}\right. = \overset{\underline{\underline{D}}}{\begin{bmatrix} \underbrace{\underline{\underline{D_x}}}_{N_{fx}} & \underbrace{\underline{\underline{D_y}}}_{N_{fy}} \end{bmatrix}} \overset{\underline{g}}{\begin{bmatrix} \underline{g_x} \\ \underline{g_y} \end{bmatrix}} \begin{matrix} \}\, N_{fx} \\ \}\, N_{fy} \end{matrix}$$

Matrix sizes:

$\underline{\underline{D}}$   $N$ by $N_f$

$\underline{\underline{D_x}}$   $N$ by $N_{fx}$

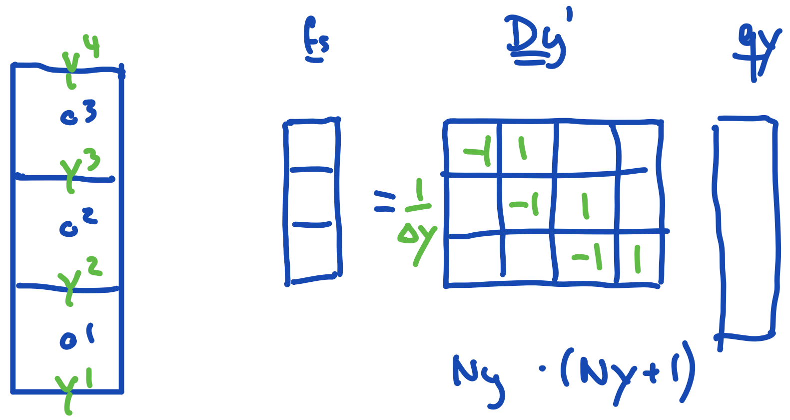
$\underline{\underline{D_y}}$   $N$ by $N_{fy}$

$\Rightarrow$ Know submatrices & shapes of $\underline{\underline{D}}$ & $\underline{G}$

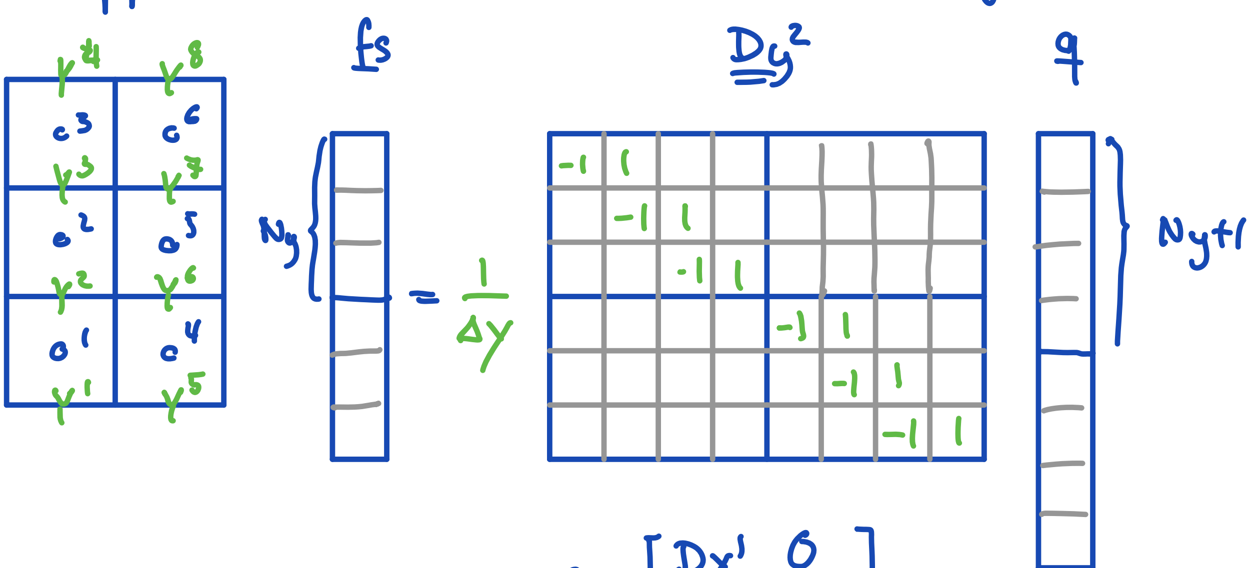
now need just the entries.

# Building the 2D discrete divergence matrix

Start with $\underline{\underline{D}}_y$ in 1D:



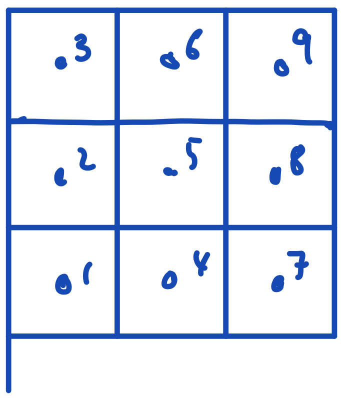

Suppose we add second column of cells




$$\underline{\underline{D}}_y^2 = \begin{bmatrix} \underline{\underline{D}}_y^1 & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{D}}_y^1 \end{bmatrix}$$

2×2 block diagonal matrix with

$\underline{\underline{D}}_y^1$ as the blocks!




$$\underline{\underline{D_y}}^2 = \begin{bmatrix} \underline{\underline{D_y}}^1 & & \\ & \underline{\underline{D_y}}^1 & \\ & & \underline{\underline{D_y}}^1 \end{bmatrix}$$

In general:

$\underline{\underline{D_y}}^2$ is a block matrix with $N_x$ by $N_x$ blocks of size $N_y$ by $(N_y+1)$ and diagonal blocks are $\underline{\underline{D_y}}^1$ and all others are zero.

## Tensor product construction of $\underline{\underline{D_y}}^2$

The discrete 2D operator can ~~be~~ easily and efficiently be assembled using kronecker/Tensor products.

Definition:

If $\underline{\underline{A}}$ is $m \times n$ matrix and $\underline{\underline{B}}$ is $p \times q$ matrix then the kronecker product

$\underline{\underline{A}} \otimes \underline{\underline{B}}$ is the $mp \times nq$ block matrix

$$\underline{\underline{A}} \otimes \underline{\underline{B}} = \begin{bmatrix} a_{11}\underline{\underline{B}} & \cdots & a_{1n}\underline{\underline{B}} \\ \vdots & & \\ a_{m1}\underline{\underline{B}} & \cdots & a_{mn}\underline{\underline{B}} \end{bmatrix}$$

Hence we can generate $\underline{\underline{D_y}}^2$ as

$$\boxed{\underline{\underline{D_y}}^2 = \underline{\underline{I_x}} \otimes \underline{\underline{D_y}}'} = \begin{bmatrix} \underline{\underline{D_y}}' & & & & & \\ & \underline{\underline{D_y}}' & & & & \\ & & \underline{\underline{D_y}}' & & & \\ & & & \underline{\underline{D_x}}'' & & \\ & & & & \underline{\underline{D_y}}' & \\ & & & & & \ddots \end{bmatrix}$$

$\underline{\underline{I_x}}$ is $N_x \cdot N_x$ identity

In Matlab the tensor product is obtained

$$\underset{\text{2D op}}{\underline{\underline{D_y}}} = \text{kron}(\underline{\underline{I_x}}, \underset{\text{1D op}}{\underline{\underline{D_y}}});$$

So how do we build $\underline{\underline{D_x}}$?

On x-first grid: $\underline{\underline{D_x}}^2 = \underline{\underline{I_y}} \otimes \underline{\underline{D_x}}'$

We'll work through this in detail next time

but the answer is $\underline{\underline{D_x}} = \underline{\underline{D_x}} \otimes \underline{\underline{I_y}}$