Lecture 16: 2D Discrete Operators – Part 2

Logistics: – Issue in HW 5 problem 2

→ will fix asap

Last time: – Intro to 2D numerics

– Matlab basics: – meshgrid & reshape

3	6	9	12
2	5	8	11
1	4	7	10

$\Rightarrow$ grid is ordered y-first

$N_f \gg N$

– $dh = \begin{bmatrix} dh_x \\ dh_y \end{bmatrix} \Rightarrow G = \begin{bmatrix} G_x \\ G_y \end{bmatrix}$ $\quad dh = G\,h$

G is N_f by N $\qquad$ D is N by N_f

– $q = \begin{bmatrix} q_x \\ q_y \end{bmatrix} \Rightarrow D = [D_x, D_y]$ $\quad D\,q = f_s$

– $D_y^2 = \begin{bmatrix} D_y' & & & \\ & D_y' & & \\ & & D_y' & \\ & & & \ddots \end{bmatrix} = I_x \otimes D_y' = \mathrm{kron}(I_x, D_y)$

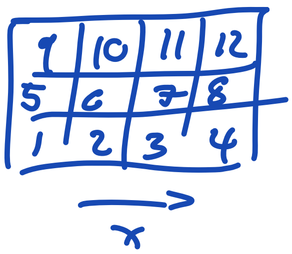
Today: – Finish the construction of D & G

– 2D advection matrix

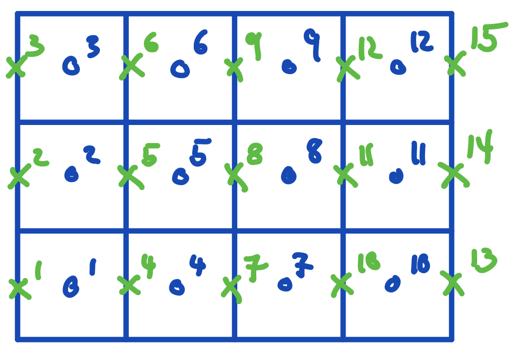
– Changes from 1D to 2D

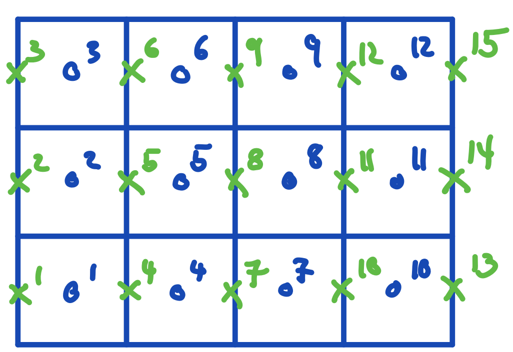
Construction of Dx

If the grid was ordered x-first



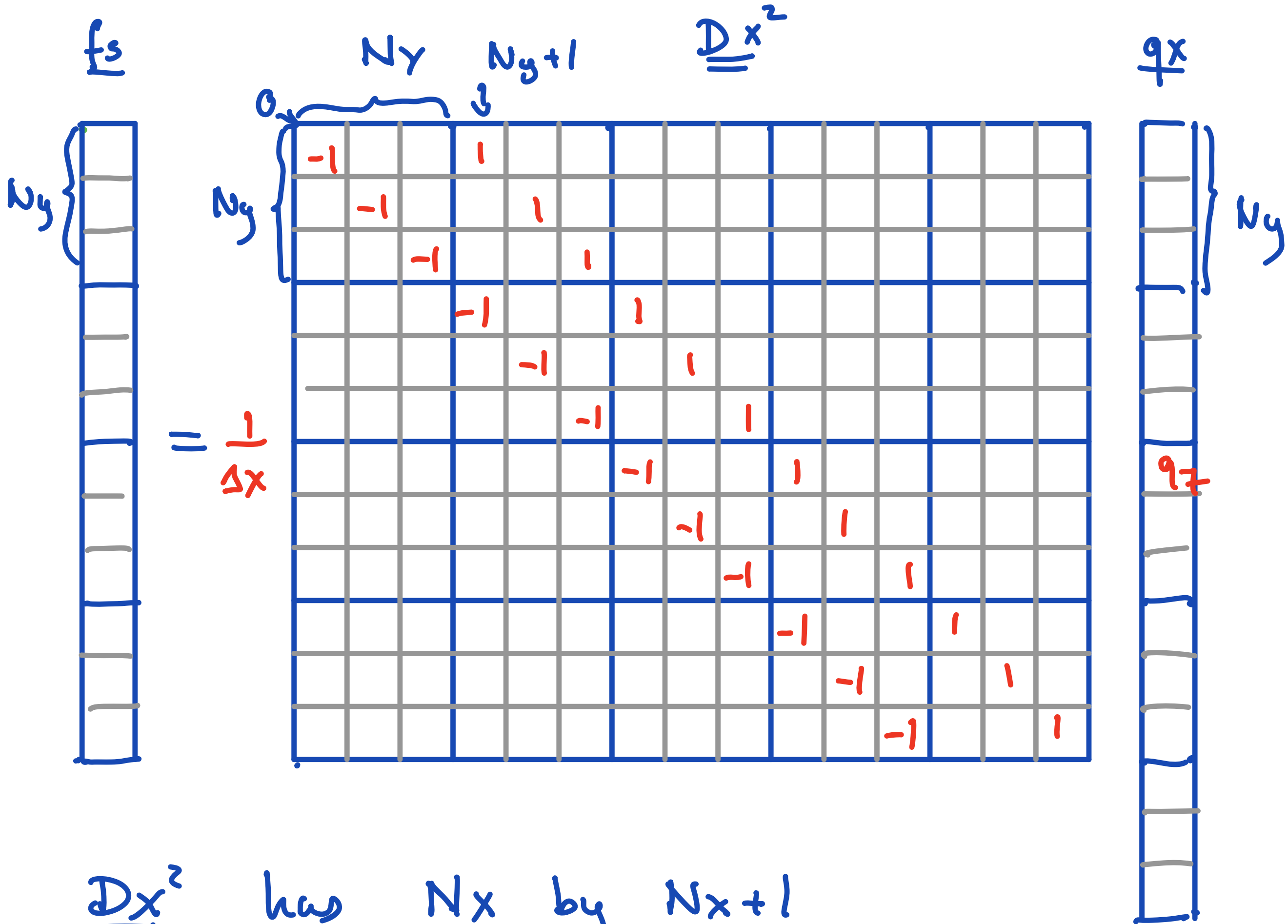
then $\underline{\underline{Dx}}^2 = \underline{\underline{Iy}} \otimes \underline{\underline{Dx}}^1$





$$f_{s_1} = \frac{q_4 - q_1}{\Delta x}$$

$$f_{s_2} = \frac{q_5 - q_2}{\Delta x}$$



$\underline{\underline{Dx}}^2$ has Nx by $Nx+1$

blocks of size Ny by Ny

Overall $\underline{\underline{Dx}}$ is N by Nfx

$\Rightarrow$ $\underline{\underline{Dx}}^2$ is a sparse diagonal matrix

that could be assembled with spdiags.

But $\underline{\underline{Dx}}^2$ is also a block diagonal matrix

built from Ny by Ny identity matrices

$$\underline{\underline{Dx}}^2 = \frac{1}{\Delta x} \begin{bmatrix} -\underline{\underline{Iy}} & \underline{\underline{Iy}} & & & \\ & -\underline{\underline{Iy}} & \underline{\underline{Iy}} & & \\ & & -\underline{\underline{Iy}} & \underline{\underline{Iy}} & \\ & & & -\underline{\underline{Iy}} & \underline{\underline{Iy}} \end{bmatrix}$$

$\underline{\underline{Iy}}$ = Ny by Ny identity

"Pattern:

$$\frac{1}{\Delta x} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix} = \underline{\underline{Dx}}'$$

$\Rightarrow$ assemble $\underline{\underline{Dx}}^2$ with tensor product

$$\underline{\underline{Dx}}^2 = \underline{\underline{Dx}}' \otimes \underline{\underline{Iy}} = \mathrm{kron}(\underline{\underline{Dx}}, \underline{\underline{Iy}})$$

In summary

Two 1D matrices : $\underline{\underline{Dx}}$, $\underline{\underline{Dy}}$

Two Identity matrices: $\underline{\underline{Ix}}$, $\underline{\underline{Iy}}$

Two 2D matrices:

$\underline{\underline{D_x}} = \text{kron}(\underline{\underline{D_x}}, \underline{\underline{I_y}});$

$\underline{\underline{D_y}} = \text{kron}(\underline{\underline{I_x}}, \underline{\underline{D_y}});$

Assemble full D:

$\underline{\underline{D}} = [\underline{\underline{D_x}}, \underline{\underline{D_y}}]$

Discrete Gradient Matrix

The $\underline{\underline{G_x}}$ and $\underline{\underline{G_y}}$ matrices can be be assembled ~~&~~ using tensor products from 1D matrices.

Instead, we use the fact that the $\underline{\underline{D}}$ and $\underline{\underline{G}}$ matrices are adjoints:

$\boxed{\underline{\underline{G}} = -\underline{\underline{D}}^T}$ true in interior

Need to impose natural BC's: set $\underline{\underline{G}} = 0$ on all bnd faces

Make vector containing all bnd faces

dof_f_bnd = [dof_f_xmin; dof_f_xmax;
dof_f_ymin; dof_f_ymax]

zero out

$\underline{\underline{G}}$(dof_f_bnd,:) = 0;

We also have $\underline{\underline{M}}$ matrix has same structure as $\underline{\underline{G}}$ matrix.

$$\underline{\underline{M}} = \begin{bmatrix} \underline{\underline{Mx}} \\ \underline{\underline{My}} \end{bmatrix}$$

generate $\underline{\underline{Mx}}^2 = \underline{\underline{Mx}}' \otimes \underline{\underline{Iy}}$ $\quad$ $\underline{\underline{My}}^2 = \underline{\underline{Ix}} \otimes \underline{\underline{My}}'$

For now we leave curl $\underline{\underline{C}}$