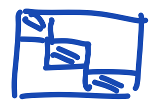
Lecture 17: 2D Discrete Advection

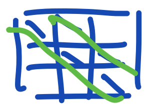
Logistics: - HW6 due Thursday

Last time: - Completed 2D ops

- $\underline{\underline{D}} = [\underline{\underline{D_x}} \;\; \underline{\underline{D_y}}]$

$$\boxed{\begin{aligned} \underline{\underline{D_y}}^2 &= \underline{\underline{I_x}} \otimes \underline{\underline{D_y}}' \\ \underline{\underline{D_x}}^2 &= \underline{\underline{D_x}}' \otimes \underline{\underline{I_y}} \end{aligned}}$$





- Discrete gradient matrix

$$\boxed{\underline{\underline{G}} = -\underline{\underline{D}}^T} \qquad \underline{\underline{G}} = 0 \text{ on bnd}$$

- Mean matrix $\underline{\underline{M}} = \begin{bmatrix} \underline{\underline{M_x}} \\ \underline{\underline{M_y}} \end{bmatrix}$

$$\underline{\underline{M_x}}^2 = \underline{\underline{M_x}}' \otimes \underline{\underline{I_y}} \qquad \underline{\underline{M_y}}^2 = \underline{\underline{I_x}} \otimes \underline{\underline{M_y}}'$$

→ Notes on 3D implementation

Today: Update advection matrix $\underline{\underline{A}}$

slightly more complicated due to variable entries

Question: 1D Melt migration model

3 PDE's

Flow problem $\phi \rightarrow \underline{q} \quad \underline{v} \quad p$

Transport problem $\rightarrow \phi$

2D discrete advection matrix

Problem: In $\underline{\underline{D}}$ and $\underline{\underline{G}}$ the matrix blocks are identical, but in $\underline{\underline{A}}$ each block has same structure but the values differ because $\underline{q}$ or $\underline{v}$ varies across the domain.

Solution: Separate the structure (1's and 0's) from the magnitudes

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\underline{q} \phi)$$

Overall scheme for assembling $\underline{\underline{A}}$ in 2D is:

$$\underbrace{\begin{bmatrix} q \\ q \end{bmatrix}}_{\underline{q}} = \underbrace{\left(\underbrace{\underline{\underline{Qd}}^{+} \begin{bmatrix} \underline{\underline{Ax}}^{+} \\ \underline{\underline{Ay}}^{+} \end{bmatrix}}_{\underline{A}^{+}} + \underbrace{\underline{\underline{Qd}}^{-} \begin{bmatrix} \underline{\underline{Ax}}^{-} \\ \underline{\underline{Ay}}^{-} \end{bmatrix}}_{\underline{A}^{-}} \right)}_{\underline{\underline{A}}^{2}} \underbrace{\begin{bmatrix} \phi \end{bmatrix}}_{\underline{\phi}}$$

where we have the following matrices:

$\underline{\underline{Qd}}^{+}$ = Nf by Nf matrix with pos. fluxes on diagonal } magnitude

$\underline{\underline{Qd}}^{-}$ = Nf by Nf matrix with neg. fluxes on diagonal }

$\underline{\underline{A}}^{+}$ = Nf by N matrix with 1's in locations of pos. fluxes

$\underline{\underline{A}}^{-}$ = Nf by N matrix with 1's in locations of neg. fluxes.

If flow is evolving only $\underline{\underline{Qd}}^{+}$ and $\underline{\underline{Qd}}^{-}$ must be recomputed ~~and~~ but $\underline{\underline{A}}^{+}$ and $\underline{\underline{A}}^{-}$ stay same.

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Qdp = spdiags(max(q,0), 0, Nf, Nf);
Qdn = spdiags(min(q,0), 0, Nf, Nf);
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So that : $\boxed{\underline{\underline{A}}(q) = \underline{\underline{Qdp}}(q)\,\underline{\underline{A_p}} + \underline{\underline{Qdn}}(q)\,\underline{\underline{A_n}}}$

where $\underline{\underline{A_p}} = \begin{bmatrix} \underline{\underline{A_{xp}}} \\ \underline{\underline{A_{yp}}} \end{bmatrix}$ $\quad \underline{\underline{A_n}} = \begin{bmatrix} \underline{\underline{A_{xn}}} \\ \underline{\underline{A_{yn}}} \end{bmatrix}$

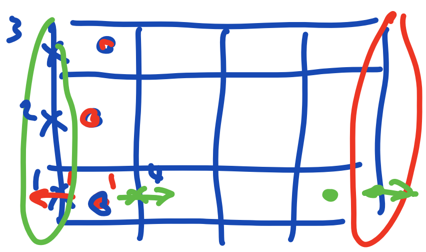
assembly of these 4 matrices is with Kronecker prod.

Ax-matrices

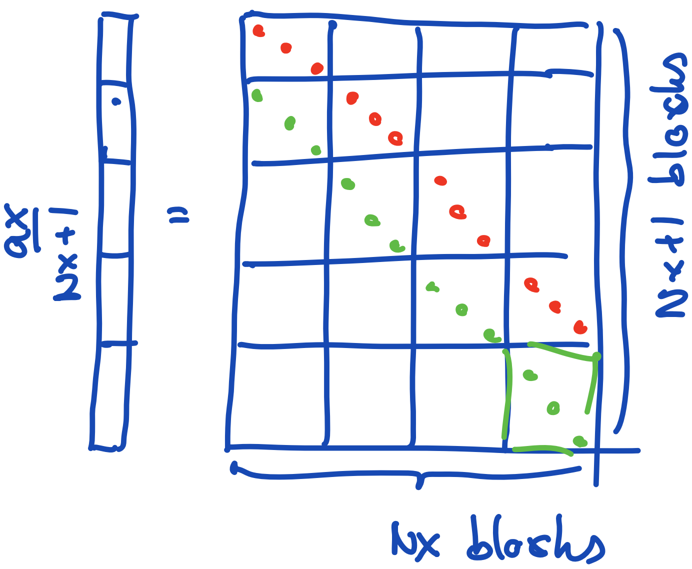
$\underline{\underline{A_x}}$ computes N_{fx} fluxes from N porosities

$N_{fx} = N_y \cdot (N_x+1)$

$N = N_y \cdot N_x$



$\underline{\underline{A_x}}$ $\underline{A_{xn}}$ $\underline{\underline{A_{xp}}}$

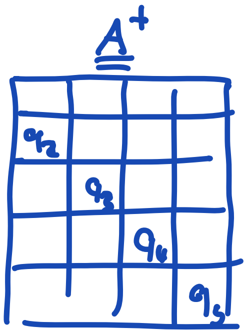


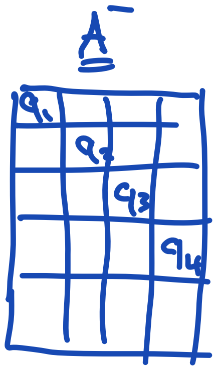
Each block is Ny by Ny

$\underline{\underline{I_y}}$ = speye(Ny)

Negative fluxes are on main diagonal

1D $\underline{\underline{A}}^+$ and $\underline{\underline{A}}^-$ matrices





The 2D matrix has same block structure as 1D matrix except without ~~ve~~ fluxes.

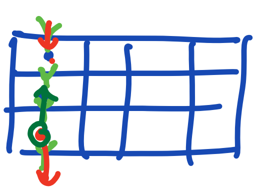
$\underline{\underline{Axp1}}$ = spdiags(ones(Nx,1), −1, Nx+1, Nx)

$\underline{\underline{Axn1}}$ = spdiags(ones(Nx,1), 0, Nx+1, Nx)

2 matrices: $\underline{\underline{Axp}}$ = kron($\underline{\underline{Axp1}}$, $\underline{\underline{Iy}}$)

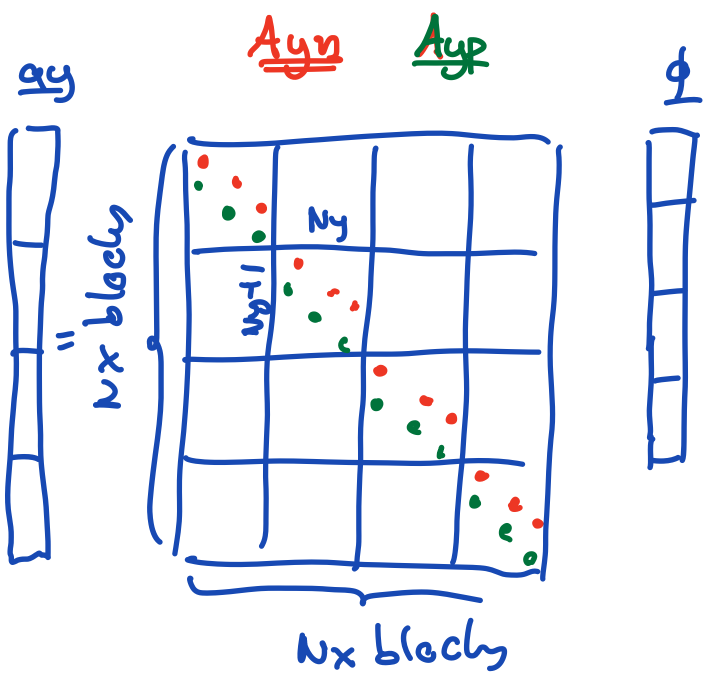
$\underline{\underline{Axn}}$ = kron($\underline{\underline{Axn1}}$, $\underline{\underline{Iy}}$)

$\underline{Ay}$ matrices



$\underline{\underline{Ay}}$ computes Nx columns of Ny+1 fluxes from Ny cell center values

⇒ $\underline{\underline{Ay}}$ is Nx by Nx block matrix with blocks of size Ny+1 by Ny



First generate 1D matrices:

$$A_{yp} = \text{spdiags}(\text{ones}(N_y,1), -1, N_y+1, N_y)$$

$$A_{yn} = \text{spdiags}(\text{ones}(N_y,1), 0, N_y+1, N_y)$$

Assemble 2D with kronecker product:

$$A_{yp} = \text{kron}(I_x, A_{yp});$$

$$A_{yn} = \text{kron}(I_x, A_{yn});$$

Assemble overall pos. and neg. matrices:

$$A_p = \begin{bmatrix} A_{xp} \\ A_{yp} \end{bmatrix} \qquad A_n = \begin{bmatrix} A_{xn} \\ A_{yn} \end{bmatrix}$$

A_{zp} $\qquad$ A_{zn}

Assemble overall 2D $\underline{\underline{A}}$:

$$A = \underline{\underline{Q_{dp}}}(q)\,\underline{\underline{A_p}} + \underline{\underline{Q_{dn}}}(q)\,\underline{\underline{A_n}}$$