Lecture 18: Navier-Stokes Equation

Logistics: - HW6 is due

- HW7 posted soon

Last time: 2D Discrete advection

$\Rightarrow$ more complicated than $\underline{G}$ $\underline{D}$

$$\boxed{\underline{A}(q) = \underline{\underline{Qdp}}(q)\,\underline{\underline{Ap}} + \underline{\underline{Qdn}}(q)\,\underline{\underline{An}}}$$

$\underline{\underline{Qdp}}$ and $\underline{\underline{Qdn}}$ are diagonal

$$\underline{\underline{Ap}} = \begin{bmatrix} \underline{\underline{Axp}} \\ \underline{\underline{Ayp}} \end{bmatrix} \qquad \underline{\underline{An}} = \begin{bmatrix} \underline{\underline{Axn}} \\ \underline{\underline{Ayn}} \end{bmatrix}$$

$\Rightarrow$ these matrices are assembled with kron

Completes 2D numerics for scalar unknowns

Today: - Navier-Stokes equations

- momentum (linear & angular)
- advective mom. flux $\rightarrow$ dyadic product
- diffusive mom. flux $\rightarrow$ stress tensor

$\Rightarrow$ Cauchy Momentum Equation

- Newtonian rheology

$\Rightarrow$ Incompressible N-S equations.

Derivation of equation of motion

Start with general conservation law:

$$\frac{\partial u}{\partial t} + \nabla \cdot \underline{j}(u) = f_s$$

u is unknown to be balanced

$j(u)$ is a set of fluxes that transport u

f_s is the set of sources & sinks of u

So far u has been a scalar $u = h$ $u = \phi$

Now we consider $\underline{u}$ to be a vector

$\Rightarrow$ balance law that is tensorial in nature

Equations of motion are based on Eulers "Principle of linear momentum":

Total force on a body is equal to the rate of change of total momentum of the body! (1752)

Hence unknown is is <u>linear</u> momentum

$\underline{u} = \rho \underline{v}$ of the body which is vector.

ρ = density $\quad \underline{v}$ = velocity

Linear momentum is generated within the body by body forces, here we consider gravity $\quad \underline{f_s} = \rho \underline{g} \quad \underline{g}$ is gravitational acceleration

Consider different fluxes of lin. momentum:

a) <u>Advective mom. flux</u>

scalar u: $\quad \underline{j_A} = \underline{v}\, u$

$$\underline{\underline{j_A}} = \underline{v} \otimes (\rho \underline{v}) = \boxed{\rho\,(\underline{v} \otimes \underline{v})}$$

here $\otimes$ is not tensor product but the outer or dyadic product

inner product: $\underline{v} \cdot \underline{v} = \underline{v}^T \underline{v} = v_i v_i = v_1 v_1 + v_2 v_2 + v_3 v_3$

outer product: $\underline{v} \otimes \underline{v} = \underline{v}\, \underline{v}^T = v_i v_j$

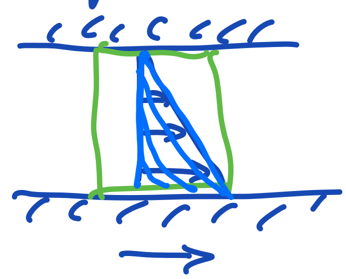
$$= \begin{pmatrix} v_1 v_1 & v_1 v_2 & v_1 v_3 \\ v_2 v_1 & v_2 v_2 & v_2 v_3 \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ v_3 v_1 & v_3 v_2 & v_3 v_3 \end{pmatrix}$$

$\Rightarrow$ advective momentum flux is inherently non-linear

b) <u>Diffusive momentum flux</u>

Linear momentum can enter a domain a control volume even if the flow is parallel to the boundary due to shear.



$$\underline{\underline{j}}_D = -\underline{\underline{\sigma}} \quad \text{where } \underline{\underline{\sigma}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}$$

$$= p\underline{\underline{I}} - \underline{\underline{\tau}}$$

$\underline{\underline{\sigma}}$ is Cauchy stress tensor, which can be decomposed into volumetric stress $-p\underline{\underline{I}}$ where p is pressure of fluid and the deviatoric stress, $\underline{\underline{\tau}}$.

Substituting into general conservation law we get Cauchy Momentum equation

$$\boxed{\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla\cdot\left[\rho(\underline{v}\otimes\underline{v}) + p\,\underline{\underline{I}} - \underline{\underline{\tau}}\right] = \rho \underline{g}}$$

This equation is very general and starting point for all fluid mechanics.

Unknowns: $\underline{v}$, p, $\underline{\underline{\tau}}$

Need a constitutive law to reduce unknowns

$$\underline{\underline{\tau}} = \underline{\underline{\tau}}(\underline{v})$$

Incompressible Newtonian Fluid

For incompressible fluid $\rho \neq \rho(p)$

In absence of changes in T or salinity

$\Rightarrow \quad \rho = \text{const.}$

Fluid mass balance: $\cancel{\frac{\partial \rho}{\partial t}}^{\,c} + \nabla\cdot(\underline{v}\rho) = 0$

$\Rightarrow \boxed{\nabla\cdot\underline{v} = 0}$ continuity eqn.

In a Newtonian fluid the deviatoric stress $\underline{\underline{\tau}}$ depends <u>linearly</u> on the strain rate $\dot{\underline{\underline{\varepsilon}}}$ $\Rightarrow$ $\boxed{\underline{\underline{\tau}} = 2\mu\, \dot{\underline{\underline{\varepsilon}}}}$

μ = dynamic viscosity $\left[\frac{M}{LT}\right]$

Rate of strain tensor: $\boxed{\dot{\underline{\underline{\varepsilon}}} = \frac{1}{2}\left(\nabla \underline{v} + \nabla^T \underline{v}\right)}$

where $\nabla \underline{v} = \begin{pmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{pmatrix}$

$\frac{\partial v_1}{\partial x_3} = v_{1,3}$

$z = x_3$

$\nabla^T \underline{v} = (\nabla \underline{v})^T$ is its transpose

$\Rightarrow \dot{\underline{\underline{\varepsilon}}} = \dot{\underline{\underline{\varepsilon}}}^T$

Now we can write deviatoric stress as

$$\underline{\underline{\tau}} = \mu\left(\nabla \underline{v} + \nabla^T \underline{v}\right)$$

so that Cauchy stress in a newtonian fluid.

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mu\left(\nabla \underline{v} + \nabla^T \underline{v}\right)$$

substitute $\underline{\underline{\sigma}}$ into Cauchy Mom. Eqn

$$\frac{\partial}{\partial t}(\cancel{\rho}\underline{v}) + \nabla\cdot\left[\cancel{\rho}(\underline{v}\otimes\underline{v}) + \frac{p}{\rho}\underline{\underline{I}} - \frac{\mu}{\rho}\left(\nabla \underline{v} + \nabla^T \underline{v}\right)\right] = \cancel{\rho}\underline{g}$$

if $\rho = \text{const}$ we can divide it out

introduce $\nu = \frac{\mu}{\rho}$ kinematic viscosity

$$\frac{1}{\rho}\nabla\cdot(p\,\underline{\underline{I}}) = \frac{1}{\rho}\nabla p$$

so that $\qquad\qquad \vec{g} = -g\hat{z}$

$$\boxed{\frac{\partial \underline{v}}{\partial t} + \nabla\cdot(\underline{v}\otimes\underline{v}) = -\frac{\nabla p}{\rho} + \nabla\cdot[\nu(\nabla\underline{v} + \nabla^T\underline{v})] + \vec{g}}$$

Incompressible N-S eqn for variable viscosity

For constant ν we can further simplify diffusive mom. flux:

$$\underbrace{\nabla\cdot[\nabla\underline{v} + \nabla^T\underline{v}]}_{\text{vector}} = \underbrace{\nabla\cdot\nabla\underline{v}}_{\text{vector}} + \underbrace{\nabla(\cancelto{0}{\nabla\cdot\underline{v}})}_{\text{vector}}$$

$$= \nabla^2\underline{v} \qquad \text{"vector laplacian"}$$

Standard Navier Stokes Equation

$$\boxed{\begin{aligned} &3\quad \frac{\partial \underline{v}}{\partial t} + \nabla\cdot[\underline{v}\otimes\underline{v}] = -\frac{\nabla p}{\rho} + \nu\nabla^2\underline{v} + \underline{g} \quad \& \quad \nabla\cdot\underline{v} = 0 \\ &\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} = -\frac{\nabla p}{\rho} + \nu\nabla^2\underline{v} + \underline{g} \quad \& \quad \nabla\cdot\underline{v} = 0 \end{aligned}}$$

First line is "conservative form"

Second line $\frac{D\underline{v}}{\partial t} = \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}$

what does $\underline{v} \cdot \nabla \underline{v}$ actually mean?

4 equations : 3 mom. bal. + 1 mass bal.

4 unknowns : 3 velocity comp. + pressure

$\Rightarrow$ closed system