Lecture 19: Stokes Equation

Logistics: HW 7 is posted due next Thursday

Last time: Navier-Stokes eqns

Gen. balance law: $\frac{\partial u}{\partial t} + \nabla \cdot j(u) = f_s$

now $\underline{u} = \rho \underline{v}$ lin. mom.

adv. flux: $\underline{\underline{j}}_A = \rho\, \underline{v} \otimes \underline{v}$ dyadic

diff. flux: $\underline{\underline{j}}_D = -\underline{\underline{\sigma}} = p\underline{\underline{I}} - \underline{\underline{\tau}}$

σ = Cauchy stress tensor

τ = deviatoric stress

Cauchy's Equ of motion:

$$\boxed{\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot [\rho(\underline{v} \otimes \underline{v}) + p\underline{\underline{I}} - \underline{\underline{\tau}}] = \rho g}$$

Constitutive law: incompressible Newtonian

$\underline{\underline{\tau}} = \mu\,(\nabla \underline{v} + \nabla^T \underline{v})$ μ = viscosity

Navier stokes Equations

$$\boxed{\begin{aligned} \frac{\partial v}{\partial t} + \underline{v} \cdot \nabla \underline{v} &= -\frac{\nabla p}{\rho} + \nu \nabla^2 \underline{v} + g \\ \nabla \cdot \underline{v} &= 0 \end{aligned}}$$

Today: Stokes Equations

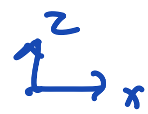
Scaling N-S Equations

$$\rho \frac{\partial \underline{v}}{\partial t} + \nabla \cdot \left[\rho\, \underline{v} \otimes \underline{v} - \mu \left(\nabla \underline{v} + \nabla^T \underline{v}\right)\right] = -\nabla p + \rho \underline{g}$$

as long as we don't consider buoyancy we can absorb the gravity term into a reduced pressure

$$\text{rhs} = -\nabla p + \rho \underline{g} = -\nabla p - \rho g \hat{\underline{z}} = -\nabla p - \rho g \nabla z = -\nabla$$

$$= -\nabla(p + \rho g z) = -\nabla \pi \qquad \hat{\underline{z}} = \nabla z$$



where $\pi = p + \rho g z$ is reduced pressure

which can be related to hyd. head $h = \frac{\pi}{\rho g}$

For now we leave N-S with red. pressure

$$\rho \frac{\partial \underline{v}}{\partial t} + \nabla \cdot \left[\rho (\underline{v} \otimes \underline{v}) - \mu \left(\nabla \underline{v} + \nabla^T \underline{v}\right)\right] = -\nabla \pi$$

Non-dimensionalization

Define generic scales

$$\underline{v}' = \frac{\underline{v}}{v_c} \qquad \underline{x}' = \frac{\underline{x}}{x_c} \qquad t' = \frac{t}{t_c} \qquad \pi' = \frac{\pi}{\pi_c} \qquad \mu' = \frac{\mu}{\mu_c}$$

substitute into mom. bal.

$$\frac{\rho v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} + \nabla'\cdot\left[\frac{\rho v_c^2}{x_c}(\underline{v}'\otimes\underline{v}') - \frac{\mu v_c}{x_c^2}\mu'(\nabla\underline{v}' + \nabla\underline{v}'^T)\right] = -\frac{\Pi_c}{x_c}\nabla'\pi'$$

First lets do an ~~adve~~ ADE scaling

→ scale to the accumulation term

$$\frac{\partial \underline{v}'}{\partial t'} + \nabla'\cdot\left[\underbrace{\frac{t_c v_c}{x_c}}_{\Pi_1}(\underline{v}'\otimes\underline{v}') - \underbrace{\frac{\mu_c t_c}{\rho x^2}}_{\Pi_2}\mu'(\nabla'\underline{v}' + \nabla'\underline{v}'^T)\right] = \underbrace{\frac{\Pi_c t_c}{\rho v_c x_c}}_{\Pi_3}\nabla'\pi'$$

Three dimensionless groups:

First two are identical to std. ADE once we recognize $\boxed{\nu = \frac{\mu_c}{\rho}}$ has units of $\frac{L^2}{T}$ and represents a "momentum diffusivity"

Π_1 and Π_2 define two time scales:

$\Pi_1 = \frac{v_c t_c}{x_c} = 1 \Rightarrow t_A = \frac{x_c}{v_c}$ advective timescale

"time to flow distance x_c with velocity v_c"

$\Pi_2 = \frac{\nu_c t_c}{x_c^2} = 1 \Rightarrow t_D = \frac{x_c^2}{\nu_c}$ diffusive time scale

"time for mom./vorticity to diffuse distance x_c"

In our applications we use Π_3 to define a pressure scale

$$\Pi_3 = \frac{\pi_c t_c}{\rho v_c x_c} = 1 \quad \Rightarrow \quad \boxed{\pi_c = \rho v_c x_c / t_c}$$

Choosing a diffusive time scale $t_c = t_D$

$$\frac{\partial v'}{\partial t'} + \nabla' \cdot \Big[\underbrace{\frac{v_c x_c}{\nu}} (\underline{v}' \otimes \underline{v}') - \mu' (\nabla' \underline{v}' + \nabla'^T \underline{v}') \Big] = \nabla' \pi'$$

One dimensionless group left. in form of a Peclet number which compares adv. & diff. mom. transport.

$$\boxed{Pe_m = \frac{v_c x_c}{\nu_c} = \frac{\rho v_c x_c}{\mu_c} = \frac{t_D}{t_A} = Re} \quad \text{Reynolds \#}$$

Dimensionless NS equation

$$\boxed{\frac{\partial v'}{\partial t'} + \nabla' \cdot \Big[Re (\underline{v}' \otimes \underline{v}') - \mu' (\nabla' v' + \nabla'^T v') \Big] = -\nabla' \pi'}$$

Clearly "momentum term" vanishes if $Re \to 0$

For application to ice, glaciers we have the following approx. parameters:

$\rho = 10^3 \quad kg/m^3$

$v_c = 1 - 100 \quad m/yr \approx 10^{-7} - 10^{-5} \frac{m}{s} \to 10^{-6} \frac{m}{s}$

$\mu_c = 10^{13} - 10^{15} \text{ Pa s} \approx 10^{14} \text{ Pa s}$

$x_c = 10^2 - 10^3 \ m \sim 10^2 \ m$

$Re = \frac{\rho v_c x_c}{\mu_c} = 10^{3-6+2-14} = 10^{-15} \ll 1$

$\Rightarrow$ adv. mom. transport is negligible.

$Re = 0$

$$\frac{\partial v'}{\partial t} - \nabla' \cdot [\mu' (\nabla' \underline{v}' + \nabla' \underline{v}'^T)] = -\nabla' \pi'$$

"Transient Stokes equ", but is it worth resolving these transients?

Estimate diff. time scale:

$t_D = \frac{x_c^2 \rho}{\mu_c} = 10^{4+3-14} s = 10^{-7} s$

Not worth resolving on timescales of 10 – 100 years

$\Rightarrow$ Choose a different ~~non-dim.~~ scaling

- clearly viscosity term is dominant
- but rate of change is instantaneous

$\Rightarrow$ scale immediately to viscous term

(dropping primes)

$$\frac{\rho v_c}{t_c}\frac{\partial v}{\partial t} + \nabla\cdot\left[\frac{\rho v_c^2}{x_c}(\underline{v}\otimes\underline{v}) - \frac{\mu_c v_c}{x_c^2}\,\mu\,(\nabla\underline{v}+\nabla^T\underline{v})\right] = -\frac{\pi_c}{v_c}\nabla\pi$$

↑ divide by this

$$\frac{\rho x_c^2}{\mu_c t_c}\frac{\partial v}{\partial t} + \nabla\cdot\left[\underbrace{\frac{v_c x_c}{\nu}}_{Re}(\underline{v}\otimes\underline{v}) - \mu\,(\nabla\underline{v}+\nabla^T\underline{v})\right] = -\underbrace{\frac{\pi_c x_c}{v_c \mu_c}}_{=1}\nabla\pi$$

$$\pi_c = \frac{\mu_c v_c}{x_c}$$

choose advective time scale: $t_c = \frac{x_c}{v_c}$

$$\frac{\rho x_c^2}{\mu_c t_c} = \frac{\rho v_c x_c}{\mu_c} = \frac{v_c x_c}{\nu_c} = Re$$

$$Re\left(\frac{\partial v}{\partial t} + \nabla\cdot(\underline{v}\otimes\underline{v})\right) - \nabla\cdot\left[\mu\,(\nabla\underline{v}+\nabla^T\underline{v})\right] = \nabla\pi$$

In limit of $Re \ll 1$ $\Rightarrow$ Stokes equation

Stokes equation

$$\boxed{\begin{aligned} -\nabla\cdot\left[\mu\left(\nabla\underline{v} + \nabla^T\underline{v}\right)\right] &= \nabla\pi \\ \nabla\cdot\underline{v} &= 0 \end{aligned}}$$

for variable viscosity

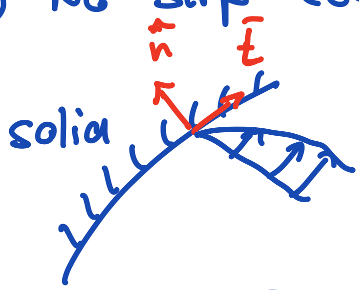
In limit of $\mu' = 1$ $\quad \mu = \text{const}$

$$\boxed{\begin{aligned} -\nabla^2\underline{v} &= \nabla\pi \\ \nabla\cdot\underline{v} &= 0 \end{aligned}}$$

Stokes flow is instantaneous

Boundary conditions for Stokes

1) No slip condition at solid bnd



if bnd is stationary $\underline{v}_b = 0$

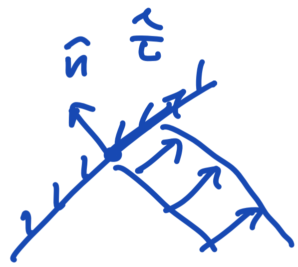
$$\boxed{\underline{v}\,\big|_{\underline{x}\in\partial\Omega} = \underline{0}}$$

Dirichlet type BC

$\Rightarrow$ prescribe velocity

$\Rightarrow$ implemented with constraints

2) Free slip BC / No shear stress



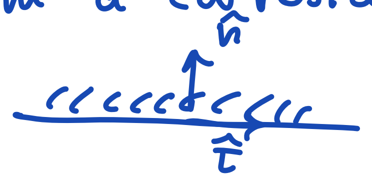
$$\underline{v}\cdot\hat{n}\big|_{\partial\Omega} = 0$$

$$\hat{\tau}\cdot(\underline{\underline{\varepsilon}}\,\hat{n})\big|_{\partial\Omega} = 0 \qquad \text{no shear stress}$$

here $\underline{t} = \underline{\underline{\varepsilon}}\,\hat{n}$ traction on bnd

$t_\parallel = \hat{\tau}\cdot\underline{t}$ comp. of traction $\parallel$ to bnd

In a cartesian geometry



$$\hat{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \hat{\tau} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{\tau}\cdot(\underline{\underline{\varepsilon}}\,\hat{n}) = (1 \quad 0)\cdot\begin{pmatrix} v_{x,x} & \frac{1}{2}(v_{x,y}+v_{y,x}) \\ \frac{1}{2}(v_{x,y}+v_{y,x}) & v_{yy} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2}\left(v_{x,y} + v_{y,x}\right)$$

since $v_y = 0$ along bnd $\frac{\partial v_y}{\partial x} = 0$

$$\Rightarrow \frac{\partial v_x}{\partial y}\Big|_{\partial\Omega} = 0 \Rightarrow \text{Neumann BC}$$