Lecture 20: Stokes grid & operators

Logistics: HW7 due next Thursday

Last time: - Stokes Equation

momentum diffusivity: $\nu = \frac{\mu}{\rho}$

Reynolds number: $Re = \frac{v_c x_c}{\nu_c} = Pe_{Mom}$

Limit $Re \to 0$ no advective mom. trans.

Stokes Equ: $\boxed{\begin{aligned} -\nabla \cdot [\mu(\nabla \underline{v} + \nabla \underline{v}^T)] &= \nabla \pi \\ \nabla \cdot \underline{v} &= 0 \end{aligned}}$

System of two equations for $\underline{v}$ & π

$\Rightarrow$ solve together

BC: No slip $\rightarrow$ Dirichlet

Free slip $\rightarrow$ Neuman/Natural.

With these velocity BC's we need

no pressure BC !

Today: Stokes Grid & operators

may take 2 lectures to complete

Discretizing Stokes Equations

Variable viscosity Stokes Eqs

1) $$\underbrace{\nabla\cdot\left[\mu\left(\nabla \underline{v} + \nabla^T \underline{v}\right)\right]}_{\underline{\underline{A}}\,\underline{v}} - \nabla p = f_1$$

2) $$\nabla\cdot\underline{v} = f_2$$

Note : $\pi \equiv p$ (easier in matlab)

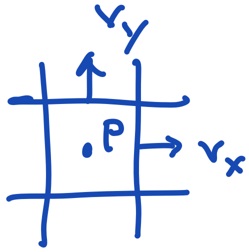
~~Choose~~ Order vectors of unknowns: $\underline{u} = \begin{bmatrix} \underline{v} \\ \underline{p} \end{bmatrix}$ discretize $\quad \underline{v} = \begin{bmatrix} \underline{v_x} \\ \underline{v_y} \end{bmatrix}$

Note Stokes is <u>linear</u> problem if $\mu \neq \mu(\underline{v}, p)$

$\Rightarrow$ we can write as discrete linear sys. of eqns.

$$\begin{bmatrix} \underline{\underline{A}} & \underline{\underline{C}}^T \\ \underline{\underline{C}} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{v} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} \underline{f}_1 \\ \underline{f}_2 \end{bmatrix}$$

We stay with same staggered grid



The discrete system

1) $$\underline{\underline{A}}\,\underline{v} + \underline{\underline{C}}^T \underline{p} = \underline{f}_1$$

2) $$\underline{\underline{C}}\,\underline{v} = \underline{f}_2$$

From 2: $\nabla \cdot \underline{v} = \underline{\underline{D}} * \underline{v} \quad \Rightarrow \quad \underline{\underline{C}} = \underline{\underline{D}}$

$\underline{\underline{D}}$ standard divergence operator

From 1: $-\nabla p \approx \underline{\underline{C}}^T \underline{p} \quad \Rightarrow \quad \nabla p = -\underline{\underline{C}}^T \underline{p}$

$$= -\underline{\underline{D}}^T \underline{p} = \underline{\underline{G}}\,\underline{p}$$

Discrete system:

$$\boxed{\begin{aligned} 1)&\quad \underline{\underline{A}}\,\underline{v} - \underline{\underline{G}}\,\underline{p} = \underline{f}_1 \\ 2)&\quad \underline{\underline{D}}\,\underline{v} = \underline{f}_2 \end{aligned}}$$

Have both $\underline{\underline{D}}$ & $\underline{\underline{G}}$

Need to form $\underline{\underline{A}}$

Divergence of deviatoric stress tensor

continuum definitions

Cauchy stress: $\underline{\underline{\sigma}} = \underbrace{-p\underline{\underline{I}}}_{\text{vol.}} + \underbrace{2\mu\dot{\underline{\underline{\varepsilon}}}}_{\underline{\underline{\tau}}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}$

Deviatoric stress: $\underline{\underline{\tau}} = 2\mu\,\dot{\underline{\underline{\varepsilon}}} = \mu\left(\nabla\underline{v} + \nabla^T\underline{v}\right)$

$$\nabla\cdot\underline{\underline{\sigma}} = \nabla\cdot\left(\underline{\underline{\tau}} - p\underline{\underline{I}}\right) = \nabla\cdot\underline{\underline{\tau}} - \underbrace{\nabla p}_{\text{done}} \qquad p = p(x)$$

$\Rightarrow$ need to discretize divergence of deviatoric stress

Definition of the div. of a 2nd order tensor:

$$\nabla \cdot \underline{\underline{\tau}} = \tau_{ij,j} \underset{\substack{\uparrow \\ \text{vector}}}{\hat{\underline{e}}_i} = \begin{pmatrix} \tau_{11,1} + \tau_{12,2} \\ \tau_{21,1} + \tau_{22,2} \end{pmatrix} = \begin{pmatrix} \nabla \cdot (\tau_{11} \;\; \tau_{12}) \\ \nabla \cdot (\tau_{21} \;\; \tau_{22}) \end{pmatrix}$$

$\Rightarrow$ Tensor divergence is applied row wise

1st row is divergence of diff mom. flux in x-dir

2nd row is divergence of diff. mom. flux in y-dir

Definition of gradient of vector:

$$\nabla \underline{v} = v_{i,j}\, \hat{e}_i \otimes \hat{\underline{e}}_j = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix} = \begin{pmatrix} \nabla v_1^T \\ \nabla v_2^T \end{pmatrix}$$

$\Rightarrow$ Gradient of vector is applied to each component

$$\nabla^T \underline{v} = (\nabla \underline{v})^T = \begin{pmatrix} v_{1,1} & v_{2,1} \\ v_{1,2} & v_{2,2} \end{pmatrix} = (\nabla v_1, \nabla v_2)$$

Rate of strain tensor:

$$\underline{\underline{\dot{\varepsilon}}} = \frac{1}{2}\left(\nabla \underline{v} + \nabla^T \underline{v}\right) = \begin{pmatrix} v_{1,1} & \frac{1}{2}(v_{1,2} + v_{2,1}) \\ \frac{1}{2}(v_{1,2} + v_{2,1}) & v_{2,2} \end{pmatrix} = \underline{\underline{\dot{\varepsilon}}}^T$$

Note: $\underline{\underline{\dot{\varepsilon}}} = \underline{\underline{\dot{\varepsilon}}}^T$ symmetric

Discretizing the strain rate tensor

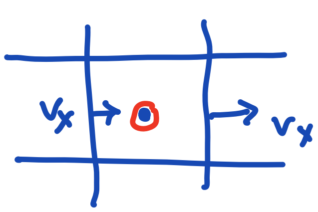
We need following derivatives

$$v_{1,1} = \frac{\partial v_1}{\partial x_1} = \frac{\partial v_x}{\partial x} \qquad v_{1,2} = \frac{\partial v_1}{\partial x_2} = \frac{\partial v_x}{\partial y}$$

$$v_{2,2} = \frac{\partial v_2}{\partial x_2} = \frac{\partial v_y}{\partial y} \qquad v_{2,1} = \frac{\partial v_2}{\partial x_1} = \frac{\partial v_y}{\partial x}$$

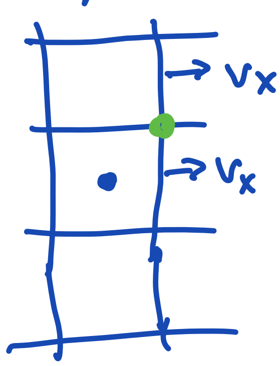
Where are these derivatives naturally approx. on our staggered mesh?

$\frac{\partial v_x}{\partial x}$



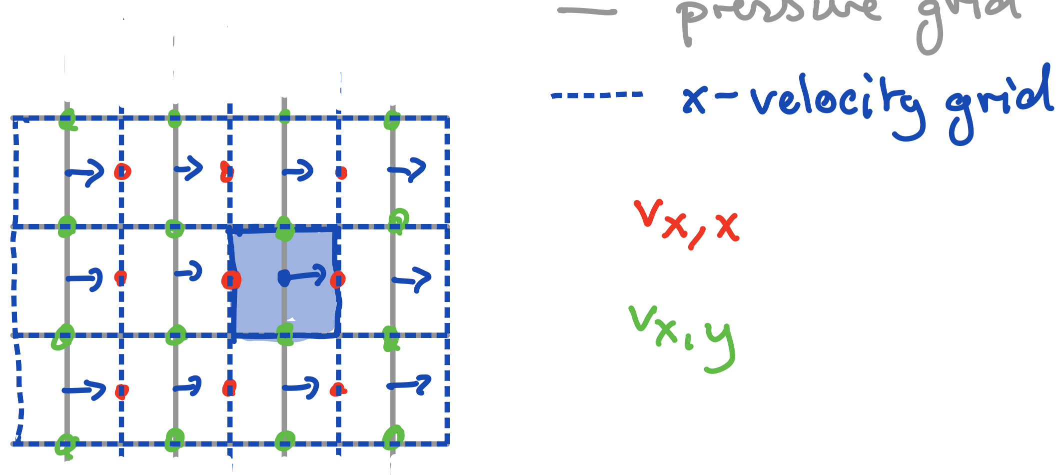
cell centers

$\frac{\partial v_x}{\partial y}$



cell corners

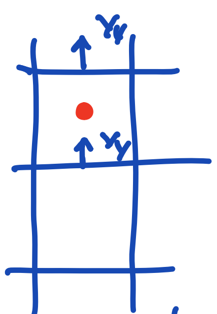
What is simplest way to compute these



Introduce a new grid that is shifted by $\frac{\Delta x}{2}$ relative to p-grid in x-direction and its size N_x+1 by N_y

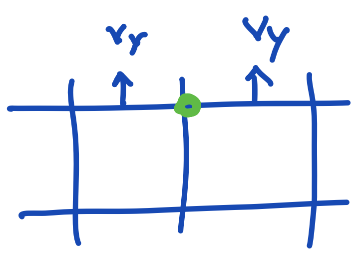
We deal with the v_y derivatives in similar way

$\frac{\partial v_y}{\partial y}$

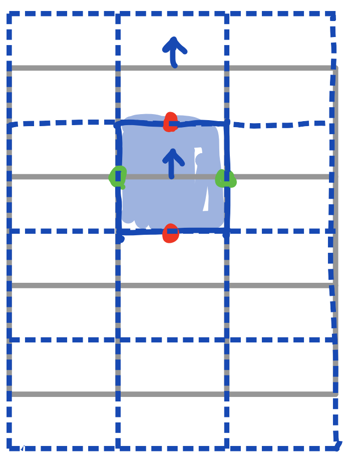


cell centre

$\frac{\partial v_y}{\partial x}$



cell corners



shifted by $\frac{\Delta y}{2}$ in y-dir

size N_x by $N_y + 1$

2D Stokes Grid

- In 2D we use 3 staggered grids

1) Pressure grid: Primary grid that defines location of pressure and velocities
   N_x by N_y
   $\Rightarrow$ use for any transport calc.

2) x-velocity grid: shifted by $\frac{\Delta x}{2}$ in x-dir relative to p-grid. Used to compute v_x derivatives in strain-rate tensor
   $N_x + 1$ by N_y

3) y-vel. grid: shifted by $\frac{\Delta y}{2}$ in y-dir
N_x by N_y+1 relative to p-grid. Used to compute v_y derivatives in strain rate tensor.

Note: In 3D we have additional z-grid

$\Rightarrow$ Need careful bookkeeping

New Matlab function:

Grid = build_stokes_grid(Gridp)

Grid.p = pressure grid
Grid.x = x-velocity grid
Grid.y = y-velocity grid

all built using standard build_grid.m