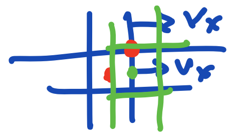
Lecture 21: Discrete Stokes Operators

Logistics: - HW7 due Thursday

Last time: - Stokes grid

- motivation: $v_{x,y}$



- 3 grids

  1) Pressure grid: - primary N_x - N_y

  2) x-velocity grid: - shifted by $\frac{\Delta x}{2}$

     N_x+1 by N_y

  3) y-velocity grid - shifted by $\frac{\Delta y}{2}$

     N_x by N_y+1

- $\underline{u} = \begin{bmatrix} \underline{v} \\ p \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ p \end{bmatrix}$

- $$\begin{aligned} \underbrace{\nabla\cdot[\mu(\nabla \underline{v} + \nabla^T \underline{v})]} - \nabla p &= f_1 \\ \nabla\cdot v &= f_2 \end{aligned}$$

discrete:

$$\underline{\underline{A}}\,\underline{v} - \underline{\underline{G}}\,\underline{p} = \underline{f}_1$$

$$\underline{\underline{D}}\,\underline{v} = \underline{f}_2$$

$$\underbrace{\begin{bmatrix} \underline{\underline{A}} & -\underline{\underline{G}} \\ \underline{\underline{D}} & \underline{\underline{0}} \end{bmatrix}}_{\underline{\underline{L}}} \underbrace{\begin{bmatrix} \underline{v} \\ \underline{p} \end{bmatrix}}_{\underline{u}} = \underbrace{\begin{bmatrix} \underline{f}_1 \\ \underline{f}_2 \end{bmatrix}}_{\underline{f}}$$

Today: we need to construct $\underline{\underline{A}}$ which computes the divergence of deviatoric stress tensor

Discretizing divergence of the deviatoric stress

$$\nabla \cdot \left[\mu \left(\nabla \underline{v} + \nabla \underline{v}^T\right)\right] = \nabla \cdot \left(2\mu \underline{\underline{\dot{\varepsilon}}}\right) \approx \underbrace{\underline{\underline{D}} * 2\mu * \underline{\underline{\underline{Edot}}}}_{\underline{\underline{A}}} * \underline{v}$$

note: standard $\underline{\underline{D}} = \underline{\underline{Dp}}$

here tensor divergence $\nabla \cdot \approx \underline{\underline{D}}$

rate of strain tensor $\underline{\underline{\dot{\varepsilon}}} \approx \underline{\underline{\underline{Edot}}}$

Discrete representation of rate of strain tensor

$$\underline{\underline{\dot{\varepsilon}}} = \begin{pmatrix} v_{x,x} & \frac{1}{2}(v_{x,y} + v_{y,x}) \\ \frac{1}{2}(v_{x,y} + v_{y,x}) & v_{y,y} \end{pmatrix} = \begin{pmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{c} \\ \dot{\varepsilon}_{c} & \dot{\varepsilon}_{yy} \end{pmatrix}$$

$\Rightarrow$ 3 independent quantities

How do we store the discretization of $\underline{\underline{\dot{\varepsilon}}}$ as a function across the domain?

As a vector: $\text{eps\_dot} = \begin{bmatrix} \underline{\text{eps\_dot\_xx}} \\ \underline{\text{eps\_dot\_yy}} \\ \underline{\text{eps\_dot\_c}} \end{bmatrix} = \underline{\underline{\text{Edot}}} * \underline{v}$

Here: <u>eps_dot_xx</u> is vector of all $\dot{\varepsilon}_{xx}$ values in all cells

<u>eps_dot_yy</u> is vector of all $\dot{\varepsilon}_{yy}$ values in all cells

<u>eps_dot_c</u> is vector of all $\dot{\varepsilon}_{c}$ values in all cells

Need to find the entries into $\underline{\underline{\text{Edot}}}$ matrix that compute these terms.

To build $\underline{\underline{\text{Edot}}}$ we need discrete gradients on the x & y velocity grids.

vx-grid: $\underline{\underline{G_x}} = \begin{bmatrix} \underline{\underline{G_{xx}}} \\ \underline{\underline{G_{xy}}} \end{bmatrix}$ vy-grid: $\underline{\underline{G_y}} = \begin{bmatrix} \underline{\underline{G_{yx}}} \\ \underline{\underline{G_{yy}}} \end{bmatrix}$

These 4 matrices allow us to compute all needed velocity derivatives:

$$\frac{\partial v_x}{\partial x} = v_{x,x} \approx \underline{\underline{G_{xx}}} * \underline{v_x} \qquad \frac{\partial v_x}{\partial y} = v_{x,y} \approx \underline{\underline{G_{xy}}} * \underline{v_x}$$

$$\frac{\partial v_y}{\partial x} = v_{y,x} \approx \underline{\underline{G_{yx}}} * \underline{v_y} \qquad \frac{\partial v_y}{\partial y} = v_{y,y} \approx \underline{\underline{G_{yy}}} * \underline{v_y}$$

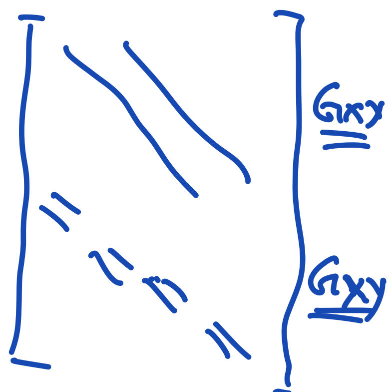
Note: build_grid gives $\underline{\underline{G_x}}$ not $\underline{\underline{G_{xx}}}$ and $\underline{\underline{G_{xy}}}$

you need to extract them

$\underline{\underline{G_{xx}}} = \underline{\underline{G_x}}(1:\underset{\uparrow}{Nfx}, :)$

Nfx for the x-velocity grid

$\underline{\underline{G_x}}$ =



$\underline{\underline{G_{xy}}} = \underline{\underline{G_x}}(\underset{\uparrow}{Nfx}+1 : Nf, :)$

from x-velocity grid.

We need to compute $\underline{\underline{\text{eps\_dot}}} = \underline{\underline{\text{Edot}}} * \underline{\underline{v}}$

$$\begin{array}{l} v_{x,x} \\ v_{y,y} \\ \frac{1}{2}(v_{x,y}+v_{y,x}) \end{array} \quad \begin{bmatrix} \text{eps\_dot\_xx} \\ \text{eps\_dot\_yy} \\ \text{eps\_dot\_c} \end{bmatrix} = \underbrace{\begin{bmatrix} G_{xx} & 0 \\ 0 & G_{yy} \\ \frac{1}{2}G_{xy} & \frac{1}{2}G_{yx} \end{bmatrix}}_{\text{Edot}} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

Now we just need the size of the zero blocks

$$\boxed{\text{Edat} = \begin{bmatrix} G_{xx} & Z_{xy} \\ Z_{yx} & G_{yy} \\ \frac{1}{2}G_{xy} & \frac{1}{2}G_{yx} \end{bmatrix}}$$

Z_{xy} is Grid.x.Nfx by Grid.y.N

Z_{yx} is Grid.y.Nfy by Grid.x.N

$\Rightarrow$ sparse allocate with spalloc

The deviatoric stress is now

$$\text{Tau} = \begin{bmatrix} \text{tau\_xx} \\ \text{tau\_yy} \\ \text{tau\_c} \end{bmatrix} = 2\mu \, \underline{\underline{\text{Edot}}} * \underline{\underline{v}}$$

To compute assembly of $\underline{\underline{A}}$ matrix we need to take divergence of $\underline{\underline{\tau}}$. To do this we need x and y submatrices of $\underline{\underline{Dx}}$ and $\underline{\underline{Dy}}$

$\underline{\underline{Dx}} = [\underline{\underline{Dxx}} \ \ \underline{\underline{Dxy}}]$ $\quad$ $\underline{\underline{Dy}} = [\underline{\underline{Dyx}} \ \ \underline{\underline{Dyy}}]$

→ extract similar to gradients

We to discretize:

$$\nabla \cdot \underline{\underline{\tau}} = \begin{bmatrix} \tau_{xx,x} + \tau_{xy,y} \\ \tau_{yx,x} + \tau_{yy,y} \end{bmatrix} \approx \underbrace{\begin{bmatrix} \underline{\underline{Dxx}} & \underline{\underline{Zyx}}^T & \underline{\underline{Dxy}} \\ \underline{\underline{Zxy}}^T & \underline{\underline{Dyy}} & \underline{\underline{Dyx}} \end{bmatrix}}_{\underline{\underline{D}}} \underbrace{\begin{bmatrix} \underline{tau\_xx} \\ \underline{tau\_yy} \\ \underline{tau\_c} \end{bmatrix}}_{\underline{tau}}$$

Hence the $\underline{\underline{A}}$ matrix is given by: $\underline{\underline{A}} = 2\mu \underline{\underline{D}} * \underline{\underline{Edot}}$

$$\underline{\underline{A}} = 2\mu \cdot \begin{bmatrix} \underline{\underline{Dxx}} & \underline{\underline{Zyx}}^T & \underline{\underline{Dxy}} \\ \underline{\underline{Zxy}}^T & \underline{\underline{Dyy}} & \underline{\underline{Dyx}} \end{bmatrix} \begin{bmatrix} \underline{\underline{Gxx}} & \underline{\underline{Zxy}} \\ \underline{\underline{Zyx}} & \underline{\underline{Gyy}} \\ \frac{1}{2}\underline{\underline{Gxy}} & \frac{1}{2}\underline{\underline{Gyx}} \end{bmatrix}$$

$$\underline{\underline{A}} = 2\mu \begin{bmatrix} \underline{\underline{D}}_{xx}\underline{\underline{G}}_{xx} + \frac{1}{2}\underline{\underline{D}}_{xy}\underline{\underline{G}}_{xy} & \frac{1}{2}\underline{\underline{D}}_{xy}\underline{\underline{G}}_{yx} \\ \frac{1}{2}\underline{\underline{D}}_{yx}\underline{\underline{G}}_{xy} & \underline{\underline{D}}_{yy}\underline{\underline{G}}_{yy} + \frac{1}{2}\underline{\underline{D}}_{yx}\underline{\underline{G}}_{yx} \end{bmatrix}$$

$\underline{\underline{A}} = \underline{\underline{A}}^T$ in the interior

We will write a function to build these operators

$[\underline{\underline{D}}, \underline{\underline{Edot}}, \underline{\underline{Dp}}, \underline{\underline{Gp}}, \underline{\underline{Z}}, \underline{\underline{I}}]$ = build_stokes_ops(Grid)

$$\underline{\underline{D}} = \begin{bmatrix} \underline{\underline{D}}_{xx} & \underline{\underline{Z}}_{yx}^T & \underline{\underline{D}}_{xy} \\ \underline{\underline{Z}}_{xy}^T & \underline{\underline{D}}_{yy} & \underline{\underline{D}}_{yx} \end{bmatrix}$$ divergence of tensor

$$\underline{\underline{Edot}} = \begin{bmatrix} \underline{\underline{G}}_{xy} & \underline{\underline{Z}}_{xy} \\ \underline{\underline{Z}}_{yx} & \underline{\underline{G}}_{yy} \\ \frac{1}{2}\underline{\underline{G}}_{xy} & \frac{1}{2}\underline{\underline{G}}_{yx} \end{bmatrix}$$ symmetric velocity gradient

$\underline{\underline{Dp}}$, $\underline{\underline{Gp}}$ are standard discrete div & grad on primary/pressure grid

$\underline{\underline{Z}}$ is an all sparse zero matrix for the lower right block of $\underline{\underline{L}}$

$$\underline{\underline{L}} = \begin{bmatrix} \underline{A} & -\underline{\underline{G}} \\ \underline{D} & \underline{\underline{Z}} \end{bmatrix}$$

$\underline{\underline{I}}$ is an $(N_f + N)$ by $(N_f + N)$ identity for the implementation of boundary conditions.