Lecture 22: Stokes BC's & Streamlines

Logistics: - HW7 is due

- HW8 will be posted (Stokes ops)

Last time: Discrete Stokes operators

$\nabla \cdot \underline{\underline{\tau}}$ divergence of deviatoric stress

$\underline{\underline{\tau}} = 2\mu \dot{\underline{\underline{\varepsilon}}} = \mu(\nabla \underline{v} + \nabla^T \underline{v})$

store $\underline{\underline{\tau}}$ and $\dot{\underline{\underline{\varepsilon}}}$ as vectors

$$\underline{\text{tau}} = \begin{bmatrix} \text{tau\_xx} \\ \text{tau\_yy} \\ \text{tau\_c} \end{bmatrix} \qquad \underline{\text{eps\_dot}} = \begin{bmatrix} \text{eps\_dot\_xx} \\ \text{eps\_dot\_yy} \\ \text{eps\_dot\_c} \end{bmatrix}$$

Discrete operator for sym. gradient

$$\underline{\underline{L}} = \begin{bmatrix} \underline{\underline{A}} & -\underline{\underline{Gp}} \\ \underline{\underline{Dp}} & \underline{\underline{Z}} \end{bmatrix} \qquad \underline{\text{eps\_dot}} = \underline{\underline{\text{Edot}}} * \underline{v}$$

Discrete operator for tensor divergence

$$\underline{\underline{D}} * \underline{\text{tau}} = \underline{f},$$

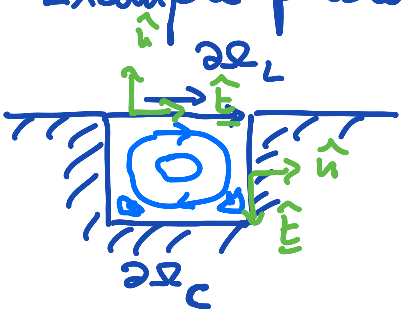
$\underline{\underline{Dp}}$ $\underline{\underline{Gp}}$

$$\underline{\underline{D}} = \begin{bmatrix} \underline{\underline{Dxx}} & \underline{\underline{Zyx}}^T & \underline{\underline{Dxy}} \\ \underline{\underline{Zxy}}^T & \underline{\underline{Dyy}} & \underline{\underline{Dyx}} \end{bmatrix} \qquad \underline{\underline{\text{Edot}}} = \begin{bmatrix} \underline{\underline{Gxx}} & \underline{\underline{Zxy}} \\ \underline{\underline{Zyx}} & \underline{\underline{Gyy}} \\ \frac{1}{2}\underline{\underline{Gxy}} & \frac{1}{2}\underline{\underline{Gyx}} \end{bmatrix}$$

$$\boxed{\underline{\underline{A}} = \underline{\underline{D}} * 2\mu * \underline{\underline{\text{Edot}}}}$$

Today: Lid-driven cavity → BC's & Streamlines

Example problem:



PDE: $\nabla \cdot [\mu(\nabla \underline{v} + \nabla^T \underline{v})] - \nabla p = 0$

$\nabla \cdot \underline{v} = 0$

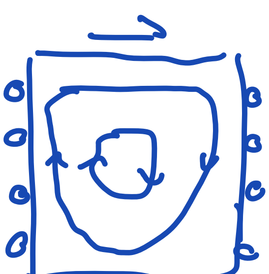
BC: $\underline{v} \cdot \hat{n} |_{\partial\Omega} = 0$ (no penetration)

$\underline{v} \cdot \hat{t} |_{\partial\Omega_C} = 0$ (no slip)

$\underline{v} \cdot \hat{t} |_{\partial\Omega_L} = V_L$ (Lid velocity)

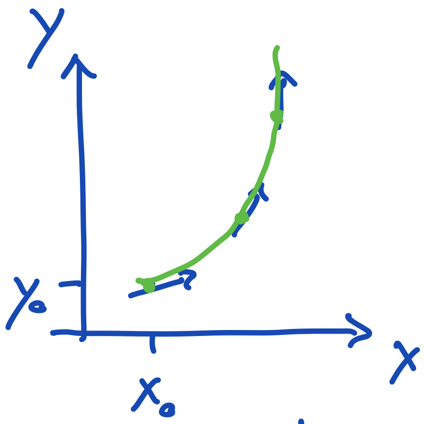
→ see live scripts

Streamlines & Streamfunctions



Streamlines provide one of best ways to visualize 2D flow fields.

Definition: Streamlines are the family of curves that are instantaneously tangent to the velocity field.

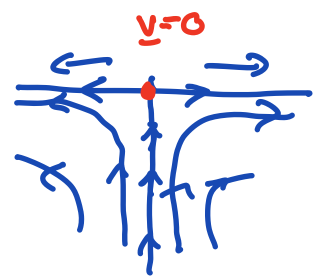


The definition of velocity provides a system of ODE's to compute streamlines

$$\frac{d\underline{x}}{dt} = \underline{v} \qquad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left.\begin{aligned} \frac{dx}{dt} &= v_x(\underline{x}) \\ \frac{dy}{dt} &= v_y(\underline{x}) \end{aligned}\right\} \quad \frac{dy}{dx} = \frac{v_y}{v_x} \qquad \underline{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

Notes:
- Safer to solve system because $v_x \to 0$
- The ODE system has problem with stagnation points

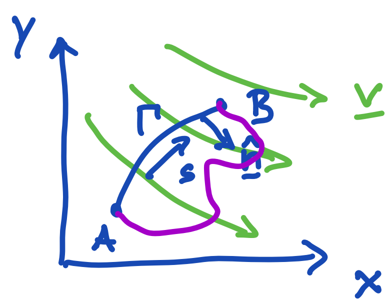


- Can only determine stagnation points by trial an error

$\Rightarrow$ but this is what streamfun.n does.

Stream function

$\rightarrow$ different way of thinking about stream lines.



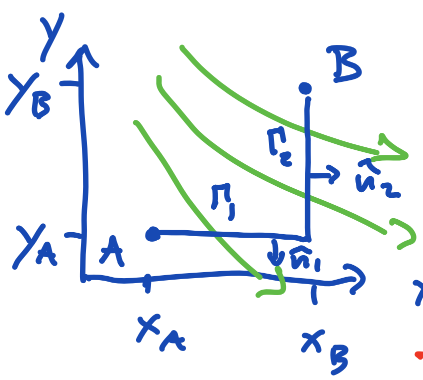
Cumulative flux between A&B

$$\boxed{\psi = \int_\Gamma \underline{v} \cdot \hat{n}\, ds}$$

Γ = path
s = arc length along Γ
$\hat{n}$ = right hand normal

In absence of fluid sources ψ should <u>not</u> depend on the path Γ.

$\Rightarrow$ choose path that simplifies integration



along Γ_1: $\underline{v} \cdot \hat{n}_1 = -v_y$ $\quad \hat{n}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

along Γ_2: $\underline{v} \cdot \hat{n}_2 = v_x$ $\quad \hat{n}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

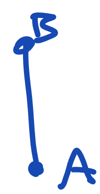
Hence we write: $$\boxed{\psi = \underbrace{\int_{x_A}^{x_B} -v_y(x, y_A)\, dx}_{\Gamma_1} + \underbrace{\int_{y_A}^{y_B} v_x(x_B, y)\, dy}_{\Gamma_2}}$$

⇒ This is definition we use to compute ψ numerically (next lecture)

Now establish relation between partials of ψ and velocity components

Suppose: $y_A = y_B$

$$\psi = \int_{x_A}^{x_B} -v_y \, dx \overset{\text{F.T.C.}}{=} \int_{x_A}^{x_B} \frac{\partial \psi}{\partial x} \, dx \Rightarrow \frac{\partial \psi}{\partial x} = -v_y$$



Suppose: $x_A = x_B$

$$\psi = \int_{y_A}^{y_B} v_x \, dy \overset{\text{F.T.C}}{=} \int_{y_A}^{y_B} \frac{\partial \psi}{\partial y} \, dy \Rightarrow \frac{\partial \psi}{\partial y} = v_x$$

Therefore: $\boxed{\dfrac{\partial \psi}{\partial x} = -v_y \quad \dfrac{\partial \psi}{\partial y} = v_x}$

This is often given as def. of stream funct.