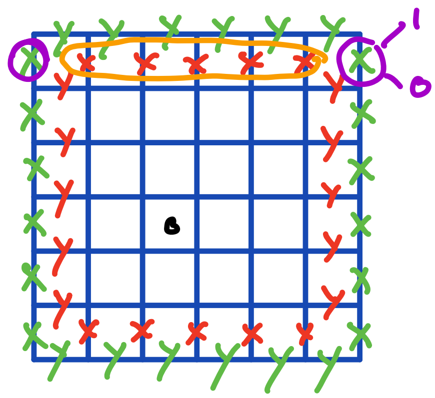
Lecture 23: Streamfunction & Streamlines

Logistics: - HW 8 is due Thursday

Last time: - Implementation of Stokes BC



normal velocities
→ (no) penetration BC's

tangential velocities
→ (no) slip BC's

pressure constraint

- need to be careful not to overspecify the BC's → exclude side faces from tangential velocity dof's

- Stream lines

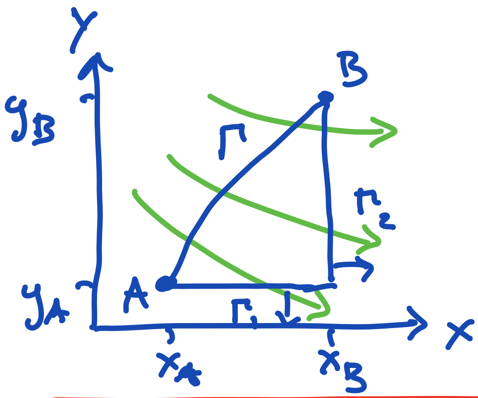
System ODE's: $\frac{d\underline{x}}{dt} = \underline{v}(\underline{x})$ $\leftarrow t$ $\qquad \frac{dy}{dx} = \frac{v_y}{v_x}$

Stream function: $\psi = \int_\Gamma \underline{v} \cdot \hat{n}\, ds$

Today: - Complete discussion of stream function

- Numerical computation of stream function

Stream function



$$\psi = \int_\Gamma \underline{v} \cdot \underline{\hat{n}} \, ds$$

should not depend on path

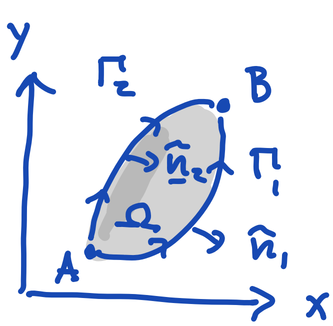
$\Rightarrow$ pick simplest path

$$\boxed{\psi = \int_{x_A}^{x_B} - v_y(x, y_A)\, dx + \int_{y_A}^{y_B} v_x(x_B, y)\, dy}$$

From this and FTC we have shown

$$\boxed{\frac{\partial \psi}{\partial x} = -v_y \qquad \frac{\partial \psi}{\partial y} = v_x}$$

This conclusion holds *if* integral is path indendent.



If path independent

$$\int_{\Gamma_1} \underline{v} \cdot \hat{n}_1 \, ds = \int_{\Gamma_2} \underline{v} \cdot \underline{n_2} \, ds$$

$$\Rightarrow \int_{\Gamma_1} v \cdot \hat{n}_1 \, ds - \int_{\Gamma_2} \underline{v} \cdot \hat{n}_2 \, ds = 0$$

Combine the path $\Gamma = \Gamma_1 + \tilde{\Gamma}_2$ but both

Γ_1 & Γ_1 are from A to B, so flip Γ_2 to

go from B to A $\Rightarrow$ changes sign and normal

$\hat{n} = \hat{n}_1$ along Γ_1 $\quad \hat{n} = -\hat{n}_2$ along $\tilde{\Gamma}_2$

$$\int_{\Gamma_1} \underline{v} \cdot \hat{n} \, ds + \int_{\tilde{\Gamma}_2} \underline{v} \cdot \hat{\underline{n}} \, ds = \oint_{\Gamma} \underline{v} \cdot \hat{n} \, ds = 0$$

Integral is path independent if $\oint_{\Gamma} \underline{v} \cdot \hat{n} \, ds = 0$

Using divergence Thm: $\oint_{\Gamma} \underline{v} \cdot \hat{n} \, ds = \int_{\Omega} \nabla \cdot \underline{v} \, dA = 0$

Integral is path independent and stream function is well defined if $\nabla \cdot \underline{v} = 0$

- flow is incompressible
- no sources/sinks in Ω

(• discussion has been in 2D)

If first two hold there is a set of two stream-functions in 3D.

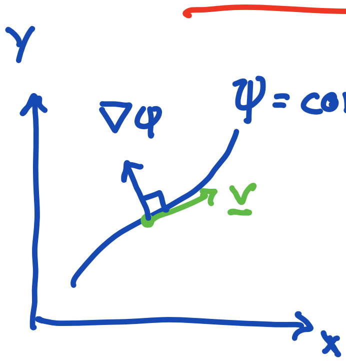
In an incompressible flow without sources/sinks the cumulative flux ψ is a single valued function $\underline{x}$ and called the streamfunction.

What is the relation between stream function and the stream lines?

Relation between ψ and streamlines

1) The level sets/contours of ψ are tangential to the velocity vector everywhere.

$\Rightarrow$ Level sets of ψ are the streamlines



$$\nabla\psi^T \cdot \underline{v} = 0$$

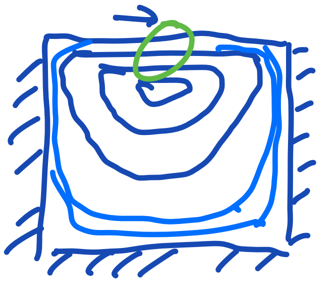
$$\begin{pmatrix} \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -v_y & v_x \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\frac{\partial\psi}{\partial x} = -v_y \qquad \frac{\partial\psi}{\partial y} = v_x$$

$$= -v_x v_y + v_x v_y = 0$$

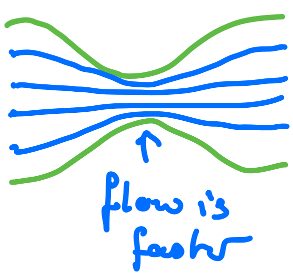
$$\Rightarrow \quad \nabla\psi \perp \underline{v}$$

2) The magnitude of velocity is equal to the magnitud of $\nabla \varphi$.



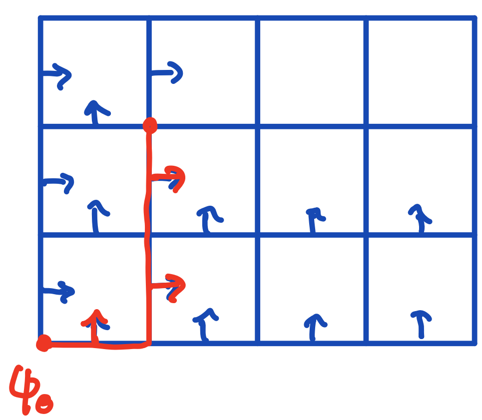
$$|\nabla \psi| = \sqrt{(-v_y)^2 + v_x^2} = \sqrt{v_x^2 + v_y^2} = |\underline{v}|$$

<u>If</u> we plot equally spaced contours of ψ then the spacind between stream lines is inversly proportiond to velocity



Computing the Streamfunction

Definition: $\psi(x,y) = \psi_0(x_0,y_0) - \int_{x_0}^{x} v_y(x',y_0)\,dx' + \int_{y_0}^{y} v_x(x,y)\,dy$



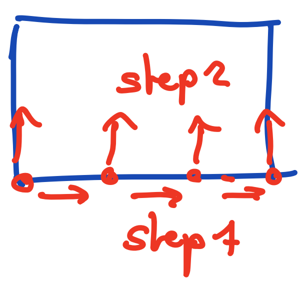
Given the location of the velocities on faces, the natural location to evaluate ψ is in the corners.

Just need to integrate v_x ~~or~~ and v_y along cell boundaries.

Implementation

- Simple Riemann sum is appropriate because $\underline{v}$ is constant along cell face
- Simple implementation with cumsum.m which ~~can~~ also works on matrices!

Step 1: Integrate along a bnd with a 1D cumsum



Step 2: First reshape the (x) velocities into appropriate matrix then you apply a ~~mat~~ 2D cumsum to simultaneously ~~sum~~ integrate all the columns/rows.

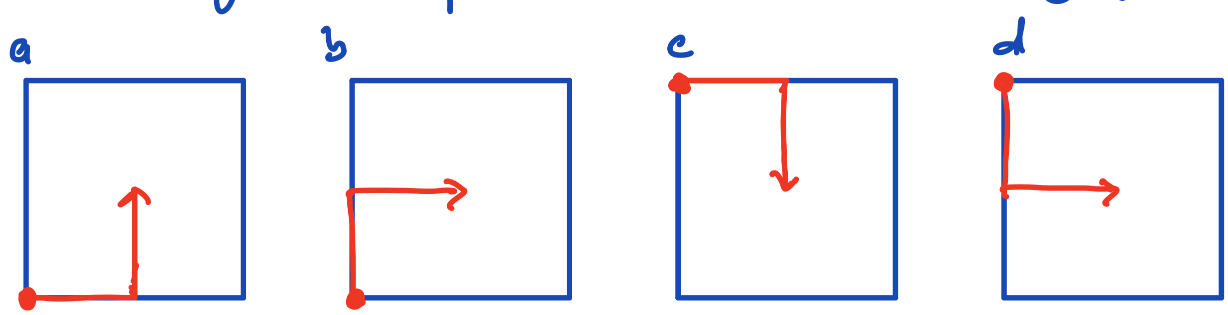
⇒ Stream function as a matrix (N_y+1) by (N_x+1) PSI

Note on path independence

ψ is single valued and uniquely defined up to a constant.

In numerical calculation we to choose

1) Starting point $\underline{x}_0$
2) integration path, ie. x-first or y-first

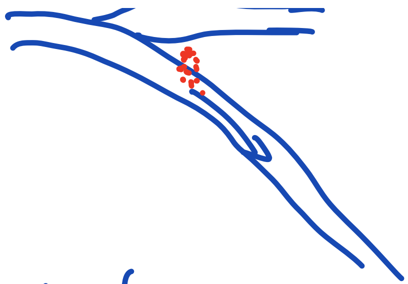


a & b must give same answer by path indep
c & d if starting point is same.

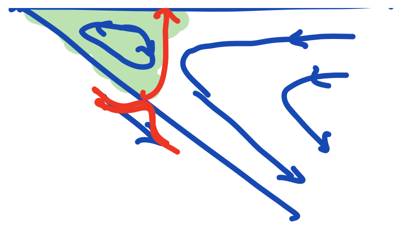
But ψ's with different starting points are offset by a constant!

Corner eddies (Keith Moffat)

Subduction zone



real



fluid mechanics

Nice idea thanks fluids guys!

But $\mu = \mu(T)$ the corner of mantle wedge is very cold $\rightarrow$ viscosity is very high

$\Rightarrow$ no corner eddy!