Lecture 24: Variable viscosity Stokes flow

Logistics: - HW 8 is due

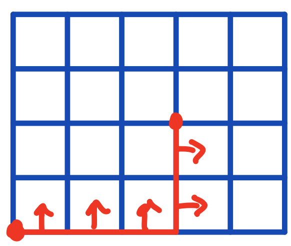
- HW 9 will be posted (Lid driven cavity)

Last time: - Streamfunction

- $\psi = -\int v_y \, dx + \int v_x \, dy$

- ψ (is single valued) if $\nabla \cdot \underline{v} = 0$

- Numerical implementation



ψ is on cell corners

simply sum up fluxes along faces

Today: - Variable viscosity

Example of Couette flow with T-gradient.

Temperature dependent viscosity

- most common source of non-linearity is variation of viscosity with Temperature
- Ice rheology is complex because it depends on the microscopic deformation mechanism
- We consider "diffusion creep" which results in Newtonian rheology

$$\boxed{\mu = \frac{R T d^2}{42 V_m D_{0,v}} \exp\left(\frac{E_A}{RT}\right)}$$

Parameters:
d = grain diameter $\sim 1\,mm$

T = temp.

V_m = molar volume $1.97 \cdot 10^{-5} \frac{m^3}{mol}$

$D_{0,v}$ = vol. diff. const $9.1 \cdot 10^{-4} \frac{m^2}{s}$

E_A = activation energy $60 \frac{kJ}{mol}$

R = univ. gas const. $8.3 \ldots \frac{J}{K\,mol}$

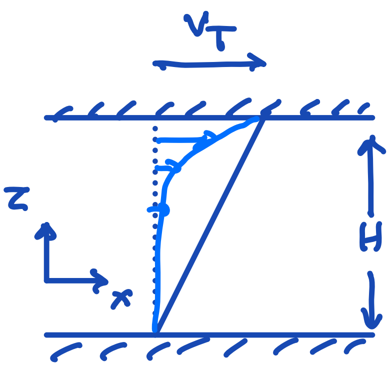
Newtonian : $\mu \neq \mu(\underline{v})$

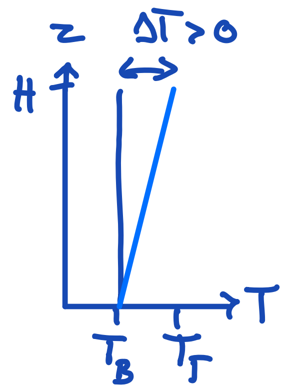
but μ has Arrhenius dependence on T

If T-dependence in pre-factor is neglected

$$\boxed{\mu = \mu_0 \exp\left(\frac{E_A}{RT}\right)} \qquad \mu_0 = \frac{R T_m d_0^2}{42 V_m D_{0,V}}$$

<u>Example problem: Couette flow with T-gradient</u>





A boundary layer forms near the hot top boundary where high T reduces μ.

In absence of viscous dissipation the T-field is independent of the velocity.

$\Rightarrow$ one way coupling: $\underline{v} = \underline{v}(T)$ but $T \neq T(\underline{v})$

Temperature field:

$$\nabla \cdot [\underline{v} T - D \nabla T] = 0 \rightarrow \frac{d^2 T}{dz^2} = 0 \Rightarrow T = T_B + \frac{\Delta T}{H} z$$

$$\boxed{\frac{T}{T_B}} = 1 + \frac{\Delta T}{T_B} \frac{z}{H} = \boxed{1 + b z'} \qquad z' = \frac{z}{H} \quad b = \frac{\Delta T}{T_B}$$

Velocity & pressure fields

$$-\nabla\cdot[\mu(T)\,(\nabla\underline{v}+\nabla^T\underline{v})]+\nabla\pi=0 \qquad \pi=p+\rho g z$$

$$\nabla\cdot\underline{v}=0$$

clearly $v_z=0 \rightarrow v_{z,x}=v_{z,z}=0$

Continuity: $\cancelto{0}{\frac{\partial v_x}{\partial x}}+\cancelto{0}{\frac{\partial v_z}{\partial z}}=0 \Rightarrow v_{x,x}=0$

Deviatoric stress tn components

$$\underline{\underline{\tau}}=\mu(\nabla\underline{v}+\nabla^T\underline{v})=\mu\begin{bmatrix}2\cancel{v_{x,x}} & v_{x,z}+\cancelto{0}{v_{z,x}}\\ \cancelto{0}{v_{z,x}}+v_{x,z} & 2\cancelto{0}{v_{z,z}}\end{bmatrix}$$

$$\underline{\underline{\tau}}=\mu\begin{bmatrix}0 & v_{x,z}\\ v_{x,z} & 0\end{bmatrix}=\begin{bmatrix}\tau_{x,x} & \tau_{x,z}\\ \tau_{z,x} & \tau_{z,z}\end{bmatrix}$$

$$\nabla\cdot\underline{\underline{\tau}}=\begin{bmatrix}\tau_{x,xx}+\tau_{x,zz}\\ \tau_{z,xx}+\tau_{z,zz}\end{bmatrix}=\begin{bmatrix}\cancelto{0}{\frac{\partial}{\partial x}(\mu(T)0)}+\frac{\partial}{\partial z}(\mu(T)v_{x,z})\\ \underbrace{\cancelto{0}{\frac{\partial}{\partial x}(\mu(T)v_{x,z})}}+\cancelto{0}{\frac{\partial}{\partial z}(\mu(T)0)}\end{bmatrix}$$

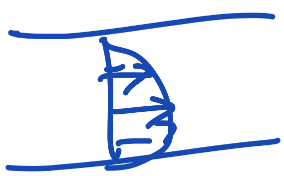
$$\nabla\cdot\underline{\underline{\tau}}=\begin{bmatrix}\frac{\partial}{\partial z}\left(\mu(T)\frac{\partial v_x}{\partial z}\right)\\ 0\end{bmatrix} \qquad \frac{\partial}{\partial x}\left[\mu(T(z))\,v_{x,z}(z)\right]=0$$

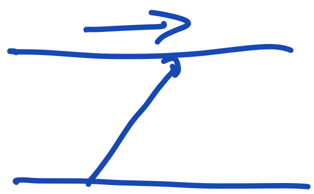
$$\nabla\pi=\begin{pmatrix}\pi_{,x}\\ \pi_{,z}\end{pmatrix}=\begin{pmatrix}\pi_{,x}\\ 0\end{pmatrix}$$ because flow is horizontal

⇒ all terms in z-momentum balance vanish.

$$\boxed{-\frac{\partial}{\partial z}\left[\mu(T(z))\frac{\partial v}{\partial z}\right]+\frac{\partial \pi}{\partial x}=0} \qquad v = v_x$$

General channel flow equation





In Pouseille (sorry) flow we have x pressure gradient

In Couette flow we don't. $\Rightarrow \frac{\partial \pi}{\partial x} = 0$

Need to solve following problem

$$\boxed{\begin{array}{ll} \text{ODE:} & \frac{d}{dz}\left[\mu(T(z))\frac{dv}{dz}\right] = 0 \\ \text{BC:} & v(0) = 0 \qquad v(H) = v_T \\ \text{Cond:} & \mu \doteq \mu_0 \exp\left(\frac{E}{RT}\right) \\ & T = T_B + \frac{\Delta T}{H} z \end{array}} \qquad E_A = E$$

Integrate once:

$$\mu \frac{dv}{dz} = c_1 \qquad \text{where} \quad c_1 = \tau = \text{shear stress}$$

Integrate again

$$v(z) = \tau \int_0^z \frac{dz'}{\mu(T(z'))} = \frac{\tau}{\mu_0} \int_0^z \exp\left(\frac{-E/R}{T(z)}\right) dz$$

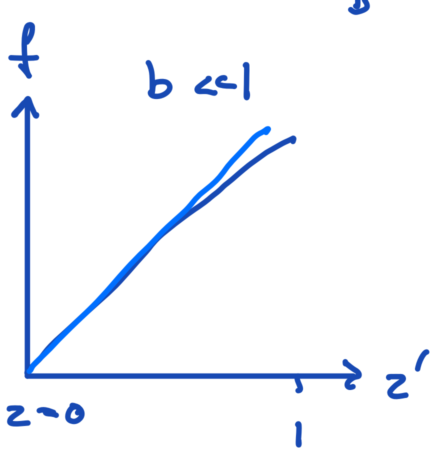
$$v(z) = \frac{\tau}{\mu_0} \int_0^z \exp\Bigg(\underbrace{\frac{-E/R}{T_B + \frac{\Delta T}{H} z}}_{f(z)}\Bigg) dz$$

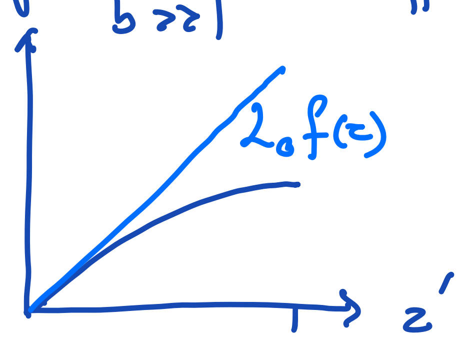
difficult integral, but if $\Delta T \ll T_B$

we can approximate exponential factor

$$f(z) = -\frac{E}{RT_B} \frac{1}{1 + \frac{\Delta T}{T_B}\frac{z}{H}} = \frac{-a}{1 + bz'} \qquad a = \frac{E}{RT_B} \quad b = \frac{\Delta T}{T_B}$$

$$z' = \frac{z}{H}$$





Almost straight line for $b \ll 1$

Approximate with Taylor series expansion

$$\alpha_0 f(z) = f(0) + \frac{df}{dz'}\Big|_0 z' = -a + ab\,z' = -a(1-bz')$$

$$\frac{df}{dz} = \frac{ab}{(1+bz')^2}$$

so we have

$$f(z) = \frac{-a}{1+bz'} \approx -a(1-bz')$$

$$e^{-a(1-bz')} = e^{-a}\, e^{+abz'}$$

$$v(z) = \frac{\tau}{\mu_0} e^{-a} \int_0^z e^{abz'} dz \qquad dz = H\,dz'$$

$$v(z') = \frac{\tau H}{\mu_0} e^{-a} \int_0^{z'} e^{abz''} dz'' = \frac{\tau H}{\mu_0} \frac{e^{-a}}{ab} \left(e^{abz'} - \underset{1}{e^0}\right)$$

$$v(z') = \frac{\tau H}{\mu_0} \frac{e^{-a}}{ab} \left(e^{abz'} - 1\right)$$

1) Set velocity → find shear stress

2) Set shear stress → find velocity

our case is (2)

$$v(z'=1) = v_T = \frac{\tau H}{\mu_0} \frac{e^{-a}}{ab} \left(e^{ab} - 1\right)$$

$$\Rightarrow \quad \tau = \frac{v_T \mu_0}{T H} \frac{ab}{e^{-a}} \frac{1}{e^{ab}-1}$$

substitute

$$\boxed{\frac{v(z')}{v_T} = \frac{e^{abz'}-1}{e^{ab}-1}}$$

where $a = \frac{E_a}{RT_B}$ $b = \frac{\Delta T}{T_B}$ $z' = \frac{z}{H}$