Lecture 26: Darcy-Stokes equations

Logistics: - still struggling with Lid-driven Cavity?

→ post ~~to~~ CDS link.

Last time: Discretizing Stokes with variable viscosity $\boxed{\mu = \mu(\underline{T})}$

$$-\nabla \cdot \left[\mu(T) \left(\nabla \underline{v} + \nabla^T \underline{v} \right) \right] + \nabla \pi = 0$$

$$\nabla \cdot \underline{v} = 0$$

$$-\underline{\underline{D}} * \underline{\underline{Mud}} * \underline{\underline{Edot}} * \underline{v} + \underline{\underline{G}}\, \underline{p} = 0$$

$$\begin{bmatrix} \text{mu\_xx} & & \\ & \text{mu\_xy} & \\ & & \text{mu\_c} \end{bmatrix} \begin{bmatrix} \text{eps\_dot\_xx} \\ \text{eps\_dot\_xy} \\ \text{eps\_dot\_c} \end{bmatrix}$$

- mu_c needs to be averaged to cell corners

$$\underline{\underline{Mc}} = \underline{\underline{Mxy}} * \underline{\underline{Mpx}} \qquad \underline{\underline{Mp}} = \begin{bmatrix} \underline{\underline{Mpx}} \\ \underline{\underline{Mpy}} \end{bmatrix} \quad \underline{\underline{Mx}} = \begin{bmatrix} \underline{\underline{Mxx}} \\ \underline{\underline{Mxy}} \end{bmatrix}$$

- The other blocks are similar to diagonals (except for "ghost points"

$$\Rightarrow \quad \underline{\underline{M_S}} = \begin{bmatrix} \underline{\underline{I_{xx}}} & & \\ & \underline{\underline{I_{xy}}} & \\ & & \underline{\underline{MC}} \end{bmatrix}$$

- Evaluate then average

  Average then evaluate

Today: Derivation of Darcy-Stokes system

Derivation of Darcy-Stokes-Equations

We have discussed flow of a porefluid

Darcy: $\nabla \cdot q = f_s \qquad q = -K \nabla h = -\frac{k}{\mu}(\nabla p + \rho g \hat{z})$

flow of a viscous fluid

Stokes: $-\nabla \cdot [\mu(T)(\nabla \underline{v} + \nabla^T \underline{v})] + \nabla \pi = 0$

And a simplified system combining the two

Helm: $-\nabla \cdot [K(\phi) \nabla h] + \frac{\phi^m}{\xi} h = \frac{\phi^m}{\xi} z$

Poisson: $-\nabla^2 u = \frac{\phi^m}{\xi}(h - z)$

Porosity: $\frac{\partial \phi}{\partial t} + \nabla \cdot (\underline{v}_s \phi) = \frac{\phi^m}{\xi}(h - z)$

Constitutive laws: $\underline{v}_s = -\nabla u$

Multiple simplyfying assumptions:

p_s = lithostatic $\qquad \underline{v}_s = -\nabla u + \cancel{\nabla \times \underline{\varphi}}$ 0

$\Rightarrow$ can be shown to be valid in small ϕ limit

without an externally imposed shear flow

To really model the problems of interest we need to be able to accomodate shear flow in the solid. $\Rightarrow$ full Darcy-Stokes eqns.

Two phase system: pore fluid (f) $\rightarrow$ melt/brine

solid matrix (s) $\rightarrow$ ice

$\phi = \phi_f$ = porosity

$1-\phi = \phi_s$ = solid vol. frac.

Mass conservation:

fluid: $\frac{\partial}{\partial t}(\cancel{\rho_f}\,\phi) + \nabla\cdot[\phi\,\underline{v}_f\,\cancel{\rho_f}] = \frac{\Gamma}{\rho_f}$

solid: $\frac{\partial}{\partial t}(\cancel{\rho_s}(1-\phi)) + \nabla\cdot[(1-\phi)\,\underline{v}_s\cancel{\rho_s}] = -\frac{\Gamma}{\rho_s}$

$\frac{\partial}{\partial t}(\phi + (1-\phi))$

we assume ρ_f and ρ_s are constant but different.

divide by densities and sum

$$\nabla\cdot[\phi\,\underline{v}_f + (1-\phi)\,\underline{v}_s] = \frac{\Gamma}{\rho_f} - \frac{\Gamma}{\rho_s} = \frac{\rho_s-\rho_f}{\rho_s\rho_f}\,\Gamma$$

introduce $\Delta\rho = \rho_f - \rho_s > 0$

$$\nabla\cdot[\phi\,\underline{v}_f + (1-\phi)\,\underline{v}_s] = -\frac{\Delta\rho}{\rho_f\rho_s}\,\Gamma$$

introduce : $\boxed{\underline{q}_r = \phi(\underline{v}_f - \underline{v}_s)} = -\frac{k}{\mu}\left(\nabla p_f + \rho_f g\,\hat{z}\right)$

eliminating $\underline{v}_f$ we get

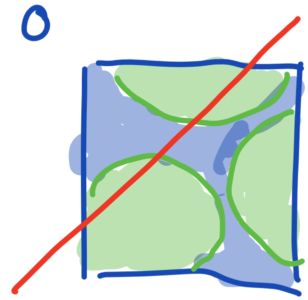
$$\boxed{\nabla\cdot[\underline{q}_r + \underline{v}_s] = -\frac{\Delta\rho}{\rho_f\rho_s}\,\Gamma}$$

Two-phase continuity

Linear momentum conservation

fluid: $\nabla\cdot[\underline{\phi}\,\underline{\underline{\sigma}}_f] - \phi\,\rho_f g\,\hat{z} + \underline{f}_I = 0$

solid: $\nabla\cdot[\underline{(1-\phi)}\,\underline{\underline{\sigma}}_s] - (1-\phi)\rho_s g\,\hat{z} - \underline{f}_I = 0$



$\underline{\underline{\sigma}}_f$ = Cauchy stress in fluid

$\underline{\underline{\sigma}}_s$ = Cauchy stress in solid

$\hat{z} = \nabla z$ where z points upwards

$\underline{f}_I$ = interaction force between solid & fluid

Note: Here we use volume fractions to represent the mean area fractions

Summing we obtain a total mom. equ.

$$\nabla \cdot [\phi \underline{\underline{\sigma}}_f + (1-\phi)\underline{\underline{\sigma}}_s] - [\phi \rho_f + (1-\phi)\rho_s] g \hat{z} = 0$$

introduce : $\bar{\rho} = \phi \rho_f + (1-\phi)\rho_s$

$$\boxed{\nabla \cdot [\phi \underline{\underline{\sigma}}_f + (1-\phi)\underline{\underline{\sigma}}_s] - \bar{\rho} g \hat{z} = 0}$$

Viscous Stress tensor

Any second-rank tensor $\underline{\underline{A}}$ can be decomposed

$$\underline{\underline{A}} = \alpha \underline{\underline{I}} + \text{dev}(\underline{\underline{A}})$$

Spherical tensor : $\alpha \underline{\underline{I}} = \frac{1}{3} \text{tr}(\underline{\underline{A}}) \underline{\underline{I}}$

Deviatoric tensor : $\text{dev}(\underline{\underline{A}}) = \underline{\underline{A}} - \alpha \underline{\underline{I}}$

$$\text{tr}(\text{dev}(\underline{\underline{A}})) = 0$$

Trace : $\text{tr}(\underline{\underline{A}}) = A_{11} + A_{22} + A_{33} = A_{ii}$

Apply spherical-deviatoric decomposition to the Cauchy stress:

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + \underline{\underline{\tau}}$$

$p = -\frac{1}{3}\,\mathrm{tr}(\underline{\underline{\sigma}})$ "mean isotropic stress" (pressure)

$\underline{\underline{\tau}} = \underline{\underline{\sigma}} - \frac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}})\,\underline{\underline{I}} = \underline{\underline{\sigma}} + p\,\underline{\underline{I}}$ deviatoric stress

Newtonian fluid: $\boxed{\underline{\underline{\tau}} = 2\mu\,\dot{\underline{\underline{\varepsilon}}}}$

$\dot{\underline{\underline{\varepsilon}}}$ = deviatoric part of rate of strain tensor

$\dot{\underline{\underline{e}}} = \frac{1}{2}\left(\nabla\underline{v} + \nabla^T\underline{v}\right)$ full rate of strain tensor

$$\dot{\underline{\underline{\varepsilon}}} = \dot{\underline{\underline{e}}} - \frac{1}{3}\,\mathrm{tr}(\dot{\underline{\underline{e}}})\,\underline{\underline{I}}$$

What is $\mathrm{tr}(\dot{\underline{\underline{e}}})$?

$$\dot{e}_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)$$

$$\mathrm{tr}(\dot{\underline{\underline{e}}}) = \dot{e}_{ii} = \frac{1}{2}\left(v_{i,i} + v_{i,i}\right) = v_{i,i} = \nabla\cdot\underline{v}$$

Substitute into $\underline{\underline{\tau}}$:

$$\underline{\underline{\tau}} = 2\mu\,\dot{\underline{\underline{\varepsilon}}} = 2\mu\left(\dot{\underline{\underline{e}}} - \frac{1}{3}\,\mathrm{tr}(\dot{\underline{\underline{e}}})\,\underline{\underline{I}}\right)$$

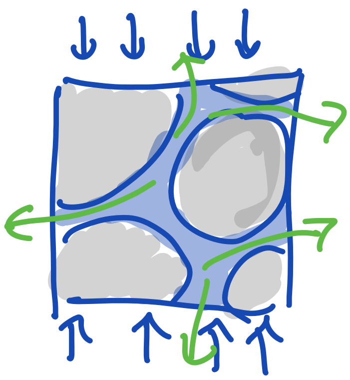
$$= \mu\left(\nabla\underline{v} + \nabla^T\underline{v} - \frac{2}{3}\,\nabla\cdot\underline{v}\,\underline{\underline{I}}\right)$$

Newtonian deviatoric stress tensor

$$\boxed{\underline{\underline{\tau}} = \mu \left(\nabla \underline{v} + \nabla^T \underline{v} - \frac{2}{3} \nabla \cdot \underline{v} \, \underline{\underline{I}} \right)}$$

So far we have considered incompressible fluid $\nabla \cdot \underline{v} = 0$

$\Rightarrow \quad \underline{\underline{\tau}} = \mu \left(\nabla \underline{v} + \nabla^T \underline{v} \right)$



Although the ice (solid) is incompressible, it's velocity field is <u>not</u> divergence free, because the mixture of ice + melt is compressible if the melt can drain!

$\Rightarrow$ need to use full deviatoric stress tensor.