Lecture 27: Darcy-Stokes II

Logistics: - sorry still didn't have time to fix HW

- CIS please fill out

Last time: - Deriving Darcy-Stokes equation

- Briefly reviewed simplified model

- Two phase mass conservation

$\nabla\cdot q_r + \nabla\cdot \underline{v}_s = 0$

$$\nabla\cdot[\phi \underline{v}_f + (1-\phi)\underline{v}_s] = \nabla\cdot[\underline{q}_r + \underline{v}_s] = -\cancel{\frac{\Delta\rho}{\rho_f\rho_s}\underline{\Gamma}} \to 0$$

- Two phase lin. mom. balance

$$\nabla\cdot[\phi\,\underline{\underline{\sigma}}_f + (1-\phi)\,\underline{\underline{\sigma}}_s] = \bar{\rho} g \hat{z}$$

- Newtonian viscous stress

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + \underline{\underline{\tau}}$$

$$p = -\tfrac{1}{3}\,\mathrm{tr}(\underline{\underline{\sigma}}) \qquad \nabla\cdot v = 0$$

$$\tau = \mu\left(\nabla\underline{v} + \nabla^T\underline{v} - \tfrac{2}{3}\,\nabla\cdot\underline{v}\,\underline{\underline{I}}\right) = 2\mu\,\underline{\underline{\dot{\varepsilon}}}$$

Today: - Compressible fluid

- Darcy stokes system

Compressible Newtonian Fluid

General compressible Cauchy stress tensor

$$\boxed{\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + 2\mu\,\dot{\underline{\underline{e}}} + \lambda\, \nabla\cdot\underline{v}\,\underline{\underline{I}}}$$

p = thermodynamic pressure (eqbm) $\rho = \rho(p)$

μ = shear viscosity

λ = second viscosity (related to compression)

Mechanical pressure / mean isotropic stress:

$$p_m = -\frac{1}{3}\,\mathrm{tr}(\underline{\underline{\sigma}}) = -\frac{1}{3}\,\mathrm{tr}\left[-p\,\underline{\underline{I}} + \lambda\,\nabla\cdot\underline{v}\,\underline{\underline{I}} + 2\mu\,\dot{\underline{\underline{e}}}\right]$$

$$= p - \lambda\,\nabla\cdot\underline{v} - \frac{2}{3}\mu\,\nabla\cdot\underline{v}$$

$$= p - \underbrace{\left(\lambda + \frac{2}{3}\mu\right)}_{\text{bulk viscosity}}\nabla\cdot\underline{v}$$

bulk viscosity $\zeta = \lambda + \frac{2}{3}\mu$

$$\boxed{p_m = p \underbrace{- \zeta\,\nabla\cdot v}_{\text{dynamic pressure}}}$$

In compressible flow the mean stress/mech. pressure differs from thermodynamic. eqbm pressure

in divergent flows $p_m < p$

Two pressures are equal if:

$\nabla\cdot\underline{v} = 0$ or $\zeta = 0$

Rewrite the Cauchy stress as:

$$\boxed{\underline{\underline{\sigma}} = -(\underbrace{p-\zeta\,\nabla\cdot\underline{v}}_{p_m})\,\underline{\underline{I}} + \mu\,(\nabla\underline{v}+\nabla^T\underline{v})}$$

lin. mom. balance:

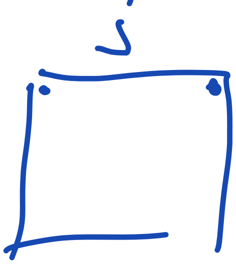
$$-\nabla\cdot\underline{\underline{\sigma}} + \rho g\hat{z} = 0 \qquad \nabla\cdot[f\,\underline{\underline{I}}] = \nabla f$$

$$-\nabla\cdot[\mu(\nabla\underline{v}+\nabla^T\underline{v})] + \nabla\cdot[(p-\zeta\,\nabla\cdot\underline{v})\,\underline{\underline{I}}] + \rho g\hat{z} = 0$$

Mass balance: $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0$

what is relation between p and ζ

$p = p_0 + \zeta(p - p_0)$ $\qquad$ p = therm. pressure



$$-\nabla\cdot[\mu(\nabla\underline{v}+\nabla^T\underline{v})] + \nabla(p-\zeta\,\nabla\cdot\underline{v}) + \rho g\hat{z} = 0$$

$\zeta = 0$: $-\nabla\cdot[\mu(\nabla\underline{v}+\nabla^T\underline{v})] + \nabla p + \rho g\hat{z} = 0$

Two-phase Darcy-Stokes

Total momentum balance:

$$\nabla \cdot \left[\phi \underline{\underline{\sigma}}_f + (1-\phi) \underline{\underline{\sigma}}_s \right] = \rho g \hat{z}$$

Need to define $\underline{\underline{\sigma}}$'s.

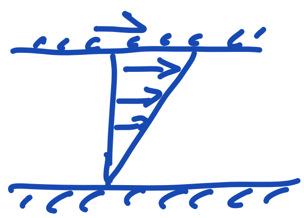
1) Stress in the pore fluid

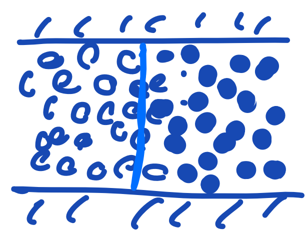
General stress tensor

$$\underline{\underline{\sigma}}_f = p_f \underline{\underline{I}} + \underline{\underline{\tau}}_f$$

Pore fluid does not accomodate deviatoric stress

$$\underline{\underline{\tau}}_f = \underline{\underline{0}}$$





$$\boxed{\underline{\underline{\sigma}}_f = - p_f \underline{\underline{I}}}$$

p_f = fluid pressure

Need to show that this reduces the lin. mom. balance in the fluid down to Darcy's law.

$$q_r = \phi (\underline{v}_f - \underline{v}_s) = - \frac{k}{\mu_f} (\nabla p_f + \rho_f g \hat{z})$$

Start with lin. mom. balance in fluid:

$$\nabla \cdot [\phi \underline{\underline{\sigma}}_f] - \phi \rho_f g \hat{z} - \underline{f}_I = 0$$

Need an expression for interaction force

$$\boxed{\underline{f}_I = c(\underline{v}_f - \underline{v}_s) - p_I \nabla\phi}$$

p_I = interface pressure

Simplest expression that is Galilean invariant.

First term is ~~is~~ viscous interaction, → drag

Second term is due to pressure acting ~~to~~ on the interface. Necessary to allow for no motion in the case of hydrostatic equilibrium.

Note: Autors differ on choice of p_I:

McKenzie (1984): $p_I = p_f$

Bercovici et al (2001): $p_I = (1-\phi)p_f + \phi p_s$

Here we choose $p_I = p_f$.

3 equs:

lin mom. bal.: $\nabla \cdot [\phi \underline{\underline{\sigma}}_f] - \phi \rho_f g \hat{z} - \underline{f}_I = 0$

Cauchy stress: $\underline{\underline{\sigma}}_f = -p_f \underline{\underline{I}}$

Interaction force: $\underline{f}_I = c(\underline{v}_f - \underline{v}_s) - p_f \nabla\phi$

$$-\underbrace{\nabla\cdot(\phi p_f \underline{\underline{I}})}_{\nabla(\phi p_f) = \phi\nabla p_f + p_f\nabla\phi} - \phi\rho_f g\hat{z} - c(\underline{v}_f - \underline{v}_s) + p_f\nabla\phi = 0$$

$$-\phi\nabla p_f - \cancel{p_f\nabla\phi} - \phi\rho_f g\hat{z} - c(\underline{v}_f - \underline{v}_s) + \cancel{p_f\nabla\phi} = 0$$

$$c(\underline{v}_f - \underline{v}_s) = -\phi(\nabla p_f + \rho_f g\hat{z})$$

$$\phi(\underline{v}_f - \underline{v}_s) = -\frac{\phi^2}{c}(\nabla p_f + \rho_f g\hat{z})$$

compare to Darcy's law:

$$\phi(\underline{v}_f - \underline{v}_s) = -\frac{k}{\mu_f}(\nabla p_f + \rho_f g\hat{z})$$

$$\Rightarrow \quad \frac{\phi^2}{c} \doteq \frac{k}{\mu_f} \qquad \boxed{c = \frac{\phi^2\mu_f}{k}}$$

$\Rightarrow$ interaction force: $\boxed{\underline{f}_I = \frac{\phi^2\mu_f}{k}(\underline{v}_f - \underline{v}_s) - p_f\nabla\phi}$

Our formulation is consistent with Darcy's law.

2) Stress in viscous fluid

General Newtonian stress tensor is

$$\underline{\underline{\sigma}}_s = -p_s\underline{\underline{I}} + \cancelto{0}{\lambda_s\nabla\cdot\underline{v}\,\underline{\underline{I}}} + 2\mu\,\dot{\underline{\underline{\varepsilon}}}_s$$

but because solid is not compressible itself

$\lambda_s = 0$ but $\nabla \cdot \underline{v}_s \neq 0$

$$\underline{\underline{\sigma}}_s = -p_s \underline{\underline{I}} + \underline{\underline{\tau}}_s$$

$$= -p_s \underline{\underline{I}} + \mu_s \left(\nabla \underline{v} + \nabla^T \underline{v} - \tfrac{2}{3} \nabla \cdot \underline{v}\, \underline{\underline{I}} \right)$$

substitute

$$\nabla \cdot \left[\phi \underline{\underline{\sigma}}_f + (1-\phi)\, \underline{\underline{\sigma}}_s \right] = \bar{\rho} g \hat{z}$$

$$\nabla \cdot \left[-\phi p_f \underline{\underline{I}} - (1-\phi) p_s \underline{\underline{I}} + (1-\phi)\mu_s \left(\nabla \underline{v} + \nabla^T \underline{v} \right) - (1-\phi)\mu_s \tfrac{2}{3} \nabla \cdot v\, \underline{\underline{I}} \right] = \bar{\rho} g \hat{z}$$

$$\mu_s^* = (1-\phi)\mu_s$$

$$\nabla \cdot \left[-(\phi p_f + (1-\phi) p_s)\, \underline{\underline{I}} - \mu_s^* \tfrac{2}{3} \nabla \cdot \underline{v}\, \underline{\underline{I}} + \mu_s^* \left(\nabla \underline{v} + \nabla^T \underline{v} \right) \right] = \bar{\rho} g \hat{z}$$

Introduce compaction relation

$$p_f - p_s = \frac{G \mu_s}{\phi^m} \nabla \cdot \underline{v}_s \qquad p_s = p_f - \frac{G \mu_s}{\phi^m} \nabla \cdot \underline{v}_s$$

$$\phi p_f + (1-\phi) p_s = \phi p_f + (1-\phi)\left(p_f - \frac{G\mu_s}{\phi^m} \nabla \cdot \underline{v}_s \right)$$

$$= \cancel{\phi p_f} + p_f - \cancel{\phi p_f} + \frac{(1-\phi)}{\phi^m} G \mu_s \nabla \cdot v_s$$

$$= p_f - \underbrace{\frac{1-\phi}{\phi^m} G \mu_s}_{\zeta_\phi} \nabla \cdot \underline{v}_s$$

$$= p_f - \zeta_\phi \nabla \cdot \underline{v}_s$$

$$\nabla \cdot \Big[-\underbrace{(p_f - \zeta_\phi \nabla \cdot \underline{v}_s)}_{\text{scalar}} \underline{\underline{I}} + \mu_s^* (\nabla \underline{v} + \nabla^T \underline{v}) \Big] = \bar{\rho} g \hat{z}$$

$$\boxed{\nabla \cdot [\mu^* (\nabla \underline{v} + \nabla \underline{v}^T)] - \nabla (p_f - \zeta_\phi \nabla \cdot \underline{v}_s) = \bar{\rho} g \hat{z}}$$

$$\boxed{\zeta_\phi = \frac{1-\phi}{\phi^m} G \mu_s}$$

$\Rightarrow$ Total momentum balance for 2 phase system has same form as ~~for~~ single phase compressible fluid mom. balance.