Lecture 28: Darcy-Stokes continued

Logistics: - All set on HW

Last time: Compressible fluid

$$-\nabla\cdot[\mu(\nabla\underline{v}+\nabla^T\underline{v})] + \nabla(p-\zeta\nabla\cdot\underline{v}) + \rho g \hat{z} = 0$$

p = thermodyn. pressure

$p_m = p - \zeta\,\nabla\cdot v$ is mech. pressure

$\zeta = \lambda + \frac{2}{3}\mu$ bulk ~~comp.~~ viscosity

- Two-phase

$\underline{\underline{\sigma}}_f = -p_f\,\underline{\underline{I}}$

$\underline{f}_I = c\,(\underline{v}_f - \underline{v}_s) - p_I\,\nabla\phi$

$\Rightarrow$ Darcy's law $\boxed{c = \frac{\phi^2\mu_f}{k}}$

$$-\nabla\cdot[\mu_s^*(\nabla\underline{v}_s+\nabla^T\underline{v}_s)] - \nabla(p_f - \zeta_\phi^*\,\nabla\cdot\underline{v}_s) + \rho_s = 0$$

$\mu_s^* = (1-\phi)\,\mu_s$

Full Darcy-Stokes Equations

1) $\nabla \cdot [q_r + \underline{v}_s] = -\frac{\Delta\rho}{\rho_f \rho_s} \Gamma$ continuity/mass

2) $\nabla \cdot [\mu_s^* (\nabla \underline{v}_s + \nabla^T \underline{v}_s - \frac{2}{3} \nabla \cdot \underline{v}_s \underline{\underline{I}})] - \nabla(p_f - \zeta_\phi \nabla \cdot \underline{v}_s) = \bar{\rho} g \hat{z}$

3) $q_r = -\frac{k}{\mu_f} (\nabla p_f + \rho_f g \hat{z})$

4) $p_f - p_s = \frac{G \mu_s}{\phi^m} \nabla \cdot \underline{v}_s$

4 equations for 4 unknowns

For single phase Stokes:

$$\nabla \cdot [\mu_s (\nabla \underline{v} + \nabla^T \underline{v})] - \nabla p = \rho g \hat{z}$$

$$\nabla \cdot \underline{v} = 0$$

$$\begin{bmatrix} \underline{\underline{A}} & -\underline{\underline{G}} \\ \underline{\underline{D}} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{v} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} f_v \\ f_p \end{bmatrix}$$

Substitute Darcy into continuity

$$\nabla \cdot \left[-\frac{k}{\mu_f} (\nabla p_f + \rho_f g \hat{z}) + \underline{v}_s\right] = -\frac{\Delta\rho}{\rho_f \rho_s} \Gamma$$

$$-\nabla \cdot \left[\frac{k}{\mu} \nabla p_f\right] + \nabla \cdot \underline{v}_s = -\frac{\Delta\rho}{\rho_f \rho_s} \Gamma + \nabla \cdot \left(\frac{k}{\mu_f} \rho_f g \hat{z}\right)$$

pg 24

Discretization of second equ.:

$$\nabla \cdot \underline{v}_s - \nabla \cdot \left[\frac{k}{\mu} \nabla p_f \right] = r.h.s.$$

$$\begin{bmatrix} \underline{\underline{A}} & -G \\ \underline{\underline{D}} & -D\,\underline{\underline{kd}}\,G \end{bmatrix} \begin{bmatrix} v_s \\ p_f \end{bmatrix} = \begin{bmatrix} f_v \\ f_p \end{bmatrix}$$

Discretization of first equs:

$$\nabla \cdot \left[\mu_s^* \left(\nabla \underline{v} + \nabla^T \underline{v} - \frac{2}{3} \nabla \cdot \underline{v}\, \underline{\underline{I}} \right) \right] - \nabla \left(p_f - \frac{1-\phi}{\phi^m} G \mu_s \nabla \cdot \underline{v}_s \right) = \bar{\rho} g \hat{z}$$

$$-\nabla \cdot \left[\frac{2}{3} (\nabla \cdot \underline{v}_s) \underline{\underline{I}} \right] = -\nabla \left(\frac{2}{3} \nabla \cdot \underline{v}_s \right)$$

$$\nabla \cdot \left[\mu_s^* \left(\nabla \underline{v} + \nabla^T \underline{v} \right) \right] - \nabla \left(p_f - \left(\frac{(1-\phi) G \mu_s}{\phi^m} - \frac{2}{3} \mu_s^* \right) \nabla \cdot \underline{v}_s \right) = \bar{\rho} g \hat{z}$$

$$\mu_s^* = (1-\phi) \mu_s \qquad \underbrace{\left(\frac{G}{\phi^m} - \frac{2}{3} \right) \mu_s^*}_{\zeta_\phi^*}$$

$$\nabla \cdot \left[\mu_s^* \left(\nabla \underline{v} + \nabla^T \underline{v} \right) \right] - \nabla \left(p_f - \zeta_\phi^* \nabla \cdot \underline{v}_s \right) = \bar{\rho} g \hat{z}$$

$$\underline{\underline{A}}\, \underline{v}_s - \underline{\underline{Gp}}\, p_f + \underline{\underline{Gp}}\, \underline{\underline{Zd}}\, \underline{\underline{Dp}}\, \underline{v}_s = \underline{f_v}$$

$$\underline{\underline{A}} = \underline{\underline{D}} + \underline{\underline{Mud}} * \underline{\underline{Edot}}$$

Zd is N by N diagonal matrix with ζ_ϕ^*

Discrete equation:

$$(\underline{\underline{A}} + \underline{\underline{G_p}}\,\underline{\underline{Z_d}}\,\underline{\underline{D_p}})\,\underline{v_s} - \underline{\underline{G_p}}\,\underline{p_f} = \underline{f_v}$$

Total discrete system:

sym?

$$\underbrace{\begin{bmatrix} \underline{\underline{A}} + \boxed{\underline{\underline{G_p}}\,\underline{\underline{Z_d}}\,\underline{\underline{D_p}}} & -\underline{\underline{G_p}} \\ \underline{\underline{D_p}} & \boxed{-\underline{\underline{D_p}}\,\underline{\underline{K_d}}\,\underline{\underline{G_p}}} \end{bmatrix}}_{\underline{\underline{L}}\,\underline{u} = \underline{f}} \underbrace{\begin{bmatrix} \underline{v_s} \\ \underline{p_f} \end{bmatrix}}_{\underline{u}} = \underbrace{\begin{bmatrix} \underline{f_v} \\ \underline{f_p} \end{bmatrix}}_{\underline{f}}$$

sym.

If ϕ, T are known then $k(\phi)$, μ_s^*, ζ_ϕ^* are given fields and we have a linear system for $\underline{v_s}$ and p_f.

In simplest case evolve porosity

$$\frac{\partial}{\partial t}(1-\phi) + \nabla\cdot[(1-\phi)\,\underline{v_s}] = -\frac{\Gamma}{\rho_s}$$

$$-\frac{\partial \phi}{\partial t} + \nabla\cdot\underline{v_s} - \nabla\cdot[\underline{v_s}\phi] = -\frac{\Gamma}{\rho_s}$$

$$\frac{\partial \phi}{\partial t} + \nabla\cdot[\underline{v_s}\phi] = +\frac{\Gamma}{\rho_s} + \nabla\cdot\underline{v_s}$$

$\Rightarrow$ porosity evolution is as before in simplified model!

To facilitate implementation of no flow BC's on the Darcy part (p_f). It would help to change from $p_f \rightarrow$ head

$$q_r = -\underline{\underline{K}} \nabla h$$

$$p = p_f - p_s$$