Lecture 29: Last Lecture

Logistics: – thats it

Last time: Darcy-Stokes system

Linear system of 2 equations: (if ϕ is given)

mom.: $\nabla\cdot\left[\mu_s^*(\nabla\underline{v}+\nabla^T\underline{v})\right]-\nabla\left(p_f-\zeta_\phi^*\nabla\cdot\underline{v}_s\right)=\bar{\rho}g\hat{z}$

mass: $-\nabla\cdot\left[\frac{k}{\mu}\nabla p_f\right]+\nabla\cdot\underline{v}_s=-\frac{\Delta\rho}{\rho_f\rho_s}\Gamma+\nabla\cdot\left(\frac{k}{\mu_f}\rho g\hat{z}\right)$

Two unknowns: $\underline{v}_s$ p_f

Discretization: $\underline{\underline{L}}\,\underline{u}=\underline{f}$

$$\underbrace{\begin{bmatrix}\underline{\underline{A}}+\underline{\underline{G_p}}\,\underline{\underline{Z_d}}\,\underline{\underline{D_p}} & -\underline{\underline{G_p}}\\ \underline{\underline{D_p}} & -\underline{\underline{D_p}}\,\underline{\underline{K_d}}\,\underline{\underline{G_p}}\end{bmatrix}}_{\underline{\underline{L}}\,\underline{u}=\underline{f}}\underbrace{\begin{bmatrix}\underline{v}_s\\ \underline{p}_f\end{bmatrix}}_{\underline{u}}=\underbrace{\begin{bmatrix}\underline{f}_v\\ \underline{f}_p\end{bmatrix}}_{\underline{f}}$$

$\underline{\underline{Z_d}}$ is N by N diagonal matrix with ζ_ϕ on the main diagonal.

Today: – Try a head formulation

Try to introduce a potential for Darcy flow because then no flow BC are natural.

Not clear what is the most useful formulation but we'll try to mimic the set up for simplified melt migration equations.

Simplified system:

overpressure: $p = p_f - p_s = p_f - p_L$

$$p_L = p_0 + \rho_s g (z_0 - z) \quad \text{lithostatic } p.$$

$$\nabla p_L = -\rho_s g \hat{z}$$

Darcy: $$q_r = -\frac{k}{\mu_f}\left(\nabla p_f + \rho_f g \hat{z}\right)$$

$$= -\frac{k}{\mu_f}\left(\nabla p_f - \nabla p_L + \nabla p_L + \rho_f g \hat{z}\right)$$

$$= -\frac{k}{\mu_f}\left(\nabla(p_f - p_L) - \rho_s g \hat{z} + \rho_f g \hat{z}\right)$$

$$= -\frac{k}{\mu_f}\left(\nabla p + \Delta\rho g \hat{z}\right) \qquad \Delta\rho = \rho_f - \rho_s > 0$$

$$\boxed{q_r = -\frac{k}{\mu_f}\left(\nabla p + \Delta\rho g \hat{z}\right)}$$

Overpressure head: $$h = z + \frac{p}{\Delta\rho g}$$

$$p = \Delta\rho g (h - z)$$

$$\nabla p = \Delta\rho g (\nabla h - \hat{z})$$

$$q_r = -\frac{k}{\mu_f}\left(\Delta\rho g (\nabla h - \hat{z}) + \Delta\rho g \hat{z}\right)$$

$$q_r = -\frac{k\Delta\rho g}{\mu_f} \nabla h = -K \nabla h$$

Substitute into total mass bal:

$$\nabla \cdot [q_r + \underline{v}_s] = -\frac{\Delta\rho}{\rho_f \rho_s}\Gamma$$

$$\boxed{-\nabla \cdot K \nabla h + \nabla \cdot \underline{v}_s = -\frac{\Delta\rho}{\rho_f \rho_s}\Gamma}$$

How does this affect total mom. balance?

$$\nabla \cdot [\phi\, \underline{\underline{\sigma}}_f + (1-\phi)\, \underline{\underline{\sigma}}_s] = \bar{\rho} g z$$

$$\underline{\underline{\sigma}}_f = -p_f \underline{\underline{I}} \qquad \underline{\underline{\sigma}}_s = -p_s \underline{\underline{I}} + \underline{\underline{\tau}}_s$$

$$\underline{\underline{\tau}}_s = \mu\left(\nabla \underline{v}_s + \nabla \underline{v}_s^T - \frac{2}{3} \nabla \cdot \underline{v}\, \underline{\underline{I}}\right)$$

$$\nabla \cdot [-\phi p_f \underline{\underline{I}} - (1-\phi) p_s \underline{\underline{I}} + (1-\phi) \underline{\underline{\tau}}_s] = \bar{\rho} g \hat{z}$$

$$\nabla \cdot \left[- \langle \underbrace{\phi p_f + (1-\phi) p_s}_{\text{term of interest}} \rangle \underline{\underline{I}} + (1-\phi) \underline{\underline{\tau}}_s \right] = \bar{\rho} g \hat{z}$$

$$\bar{p} = \phi p_f + (1-\phi) p_s$$

Aim is to a) eliminate one pressure and b) write remaining pressure term as head !

Introduce: $p = p_f - p_L$

$$\bar{p} = \phi p_f + (1-\phi) p_s - (1-\phi) p_L + (1-\phi) p_L$$

$$= \phi p_f + (1-\phi)(p_s - p_L) + (1-\phi) p_L$$

$\hat{p}_s = p_s - p_L$ solid pressure perturbation

$$= \phi p_f + (1-\phi) \hat{p}_s + (1-\phi) p_L$$

$$= \phi p_f - \phi p_L + \phi p_L + (1-\phi) \hat{p}_s + (1-\phi) p_L$$

$$= \phi (p_f - p_L) + (1-\phi) \hat{p}_s + \cancel{\phi p_L} + p_L - \cancel{\phi p_L}$$

$p = p_f - p_L$

$$\bar{p} = \phi p + (1-\phi) \hat{p}_s + p_L$$

we want to eliminate $\hat{p}_s$

Compaction relation: $p_f - p_s = \dfrac{G \mu_s}{\phi^m} \nabla \cdot \underline{v}_s$

$$p_f - p_L + p_L - p_s = p - \hat{p}_s = \frac{G\mu_s}{\phi^m} \nabla \cdot \underline{v}_s$$

$$\hat{p}_s = p - \frac{G\mu}{\phi^m} \nabla \cdot \underline{v}_s$$

substitute

$$\bar{p} = \phi p + (1-\phi)\left(p - \frac{G\mu_s}{\phi^m} \nabla \cdot \underline{v}_s\right) + p_L$$

$$= \phi p + (1-\phi) p - \underbrace{\frac{1-\phi}{\phi^m} G\mu_s}_{S_\phi} \nabla \cdot v_s + p_s$$

$$\bar{p} = p - S_\phi \nabla \cdot \underline{v}_s + p_L$$

$$h = z + \frac{p}{\Delta\rho g} \qquad p_L = \cancelto{0}{p_0} + \rho_s g (\cancelto{0}{z_e} - z)$$

$$p_L = -\rho_s g z$$

$$p = \Delta\rho g (h - z)$$

substitute

$$\bar{p} = \Delta\rho g (h - z) - S_\phi \nabla \cdot \underline{v}_s - \rho_s g z$$

$$= \Delta\rho g h - S_\phi \nabla \cdot v_s - (\rho_f - \rho_s) g z - \rho_s g z$$

$$= \qquad\qquad + (-\rho_f + \cancel{\rho_s} - \cancel{\rho_s}) g z$$

$$\bar{p} = \Delta\rho g h - \zeta_\phi \nabla\cdot \underline{v}_s - \rho_f g z$$

substitute into total mom:

$$\nabla\cdot\left[-\bar{p}\,\underline{\underline{I}} + (1-\phi)\underline{\underline{\tau}}_s\right] = \bar{\rho} g \hat{z} = (\phi\rho_f + (1-\phi)\rho_s) g \hat{z}$$

$+\rho_f g z)\underline{\underline{I}}$

$$\nabla\cdot\left[(-\Delta\rho g h + \zeta_\phi \nabla\cdot v_s + (1-\phi)\underline{\underline{\tau}}_s\right] = \phi\rho_f g \hat{z} + (1-\phi)\rho_s g \hat{z}$$

$$\nabla\cdot\left[(-\Delta\rho g h + \zeta_\phi \nabla\cdot \underline{v}_s)\underline{\underline{I}} + (1-\phi)\underline{\underline{\tau}}_s\right] = -\rho_f g \hat{z} + \phi\rho_f g \hat{z} + (1-\phi)\rho_s g \hat{z}$$

$$= -(1-\phi)\rho_f g \hat{z} + (1-\phi)\rho_s g \hat{z}$$

$$= -(1-\phi)\Delta\rho g \hat{z}$$

$$\nabla\cdot\left[(-\Delta\rho g h + \zeta_\phi \nabla\cdot \underline{v}_s)\underline{\underline{I}} + (1-\phi)\underline{\underline{\tau}}_s\right] = -(1-\phi)\Delta\rho g \hat{z}$$

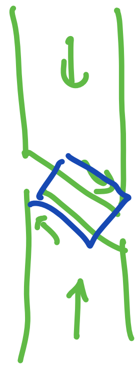
$\mu_s(1-\phi)$ $\qquad\qquad\qquad\qquad$ ξ

$$\boxed{\nabla\cdot\left[\mu_s^*(\nabla \underline{v}_s + \nabla \underline{v}_s^T)\right] - \Delta\rho g \nabla\left(h - \frac{\zeta_\phi^*}{\Delta\rho g}\nabla\cdot \underline{v}_s\right) = -(1-\phi)\Delta\rho g \hat{z}}$$

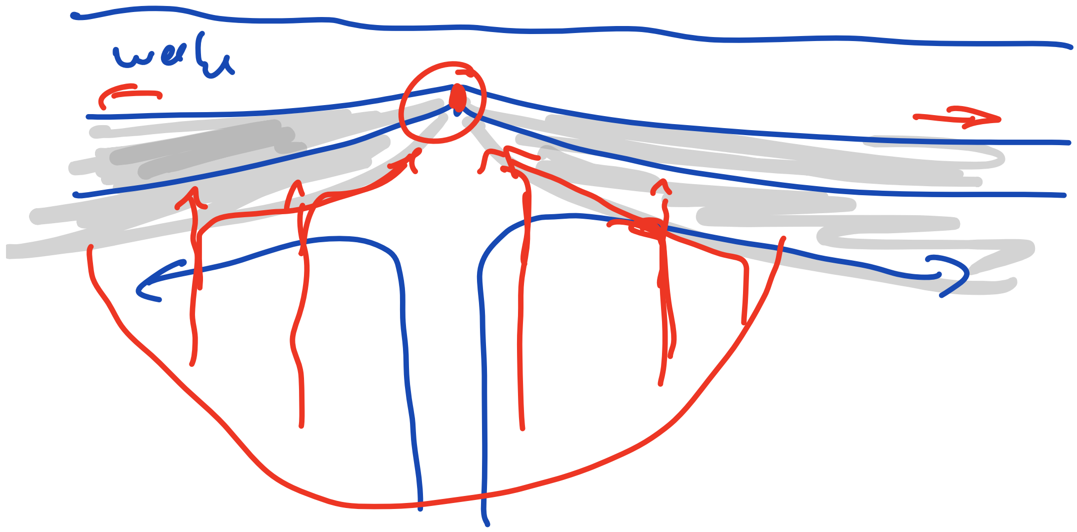
Ξ

$$\zeta_\phi^* = \frac{1-\phi}{\phi^m} G \mu_s - \frac{2}{3}\mu_s(1-\phi)$$

$$= \left(\frac{G}{\phi^m} - \frac{2}{3}\right)\mu_s(1-\phi)$$



Antithetic shear bands



The End

$$\underline{\underline{D_p}} = [\underline{D_{px}} \;\; \underline{D_{py}}]$$

$$\underline{\underline{D_p}} * v = \underline{D_{px}} * \underline{v}_x + \underline{D_{py}}\, v_y$$

$$\underline{\underline{\tau}} = \mu\left(\nabla \underline{v} + \nabla^T \underline{v}\right) - \underline{\frac{2}{3}\, \nabla\cdot\underline{v}\, \underline{\underline{I}}}$$

$$\nabla\cdot\underline{\underline{\tau}} = \nabla\cdot\left[\mu\left(\nabla \underline{v} + \nabla^T \underline{v}\right)\right] - \frac{2}{3}\mu\, \underline{\nabla(\nabla\cdot\underline{v})}$$

$$\underline{\underline{\tau}} = \underline{\underline{Tau}} \cdot \underline{v}$$

$$= \mu\, \underline{\underline{Edot}} +$$

$$\cancel{\underline{\underline{Edot}}}\ \underline{\underline{Tau}} = \begin{bmatrix} Gxx & 2 \\ 2 & Gyy \\ \frac{1}{2}Gxy & \frac{1}{2}Gyx \end{bmatrix} \begin{bmatrix} \underline{vx} \\ \underline{vy} \\ p \end{bmatrix}$$

$$\underline{\underline{Edot}}\ \underline{vx} = \underline{eps\_dot} = \begin{bmatrix} eps\_dot\_xx \\ eps\_dot\_yy \\ eps\_dot\_c \end{bmatrix}$$

$$\underline{\underline{Tau}} = \mu \begin{bmatrix} \underline{\underline{Gxx}} - \frac{2}{3}\underline{Dpx} & -\frac{2}{3}\underline{Dpy} \\ -\frac{2}{3}\underline{\underline{Dpx}} & \underline{\underline{Gyy}} - \frac{2}{3}\underline{Dpx} \\ \frac{1}{2}Gxy & \frac{1}{2}Gyx \end{bmatrix} \begin{bmatrix} vx \\ vy \end{bmatrix}$$

- $\underline{\underline{D}} \cdot q = f$

$$q = -\kappa G h$$