Lecture 2: Balance laws

Logistics: - please fill out Office hrs poll

Last time: - Course intro

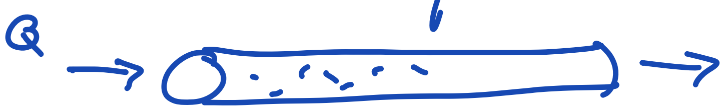
- Project: Two-phase convection
  in Europa's ice shell
- Porous media basics
  Rate, flux, velocity

Today: - General balance law

- Mass balance
- Elastic vs ductile porous media
- Incompressible flow
- hydr. head & conductivity

Balance laws

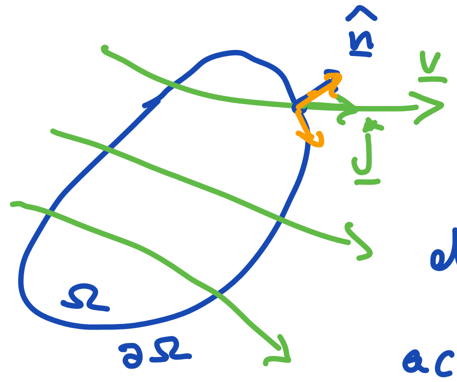
- most fund. equations → balance laws
- Balance law accounts for losses and gains of a quantity ~~due~~ within a control volume due transport or sources/sinks.
- If there are no sources/sinks then quantity is conserved → conservation law
- In multiphase systems it is not trivially obvious which quantities are conserved.



- fluid mass → conserved
- fluid energy ? not conserved

→ total energy fluid + solid is conserved

Derive a general balance law



Account for change of unknown $u(\underline{x},t)$ in domain Ω due fluxes $\underline{j}$ across boundary $\partial\Omega$ and by $\hat{f}_s$ prod. / consumption within Ω.

Units of basic quantities

- u is a density $\left[\frac{\#}{L^3}\right]$
- $\underline{j}$ is a flux $\left[\frac{\#}{L^2 T}\right]$
- $\hat{f}_s$ is a vol. rate $\left[\frac{\#}{L^3 T}\right]$

General balance on Ω:

$$\boxed{\frac{d}{dt}U = J + F}$$

1) U is total amount of u in Ω

$$U(t) = \int_\Omega u(\underline{x},t)\, \underset{L^3}{dV} \qquad [\#]$$

2) J is rate of transport of u across the total bnd $\partial\Omega$ by $\underline{j}$:

$$J(t) = -\int_{\partial\Omega} \underline{j} \cdot \hat{n} \, dA \qquad \left[\frac{\#}{T}\right]$$

L^2

outflow normal to $\partial\Omega$

3) F is total rate of prod./consump. of u in Ω

$$F = \int_{\Omega} \hat{f}_s \, dV \qquad \left[\frac{\#}{T}\right]$$

Substitute into balance equ:

$$\boxed{\frac{d}{dt}\int_{\Omega} u \, dV = -\int_{\partial\Omega} \underline{j} \cdot \hat{n} \, dA + \int \hat{f}_s \, dV}$$

Integral balance law

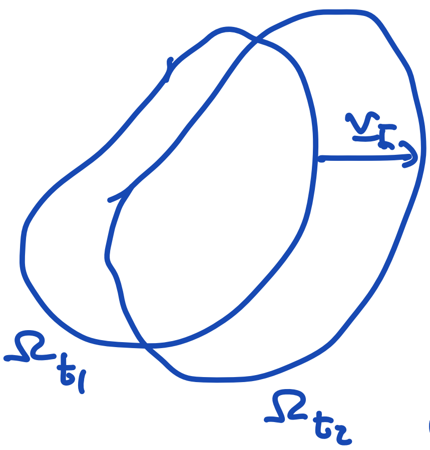
each term is a rate of change of #

To obtain a PDE we need to bring everything into same integral

$$\int_{\Omega} \cdots \, dV = 0$$

1) Exchange derivative and integral
2) Surface int → Volume.

1) Reynolds Transport Theorem



domain is moving with $\underline{v}_I$

$$\frac{d}{dt}\int_{\Omega(t)} u \, dV = \int_{\Omega} \frac{\partial u}{\partial t} \, dV + \int_{\partial\Omega} u(\underline{v}_I \cdot \hat{n}) \, dS$$

We consider Eulerian limit of fixed control volume: $\underline{v}_I = 0$

$$\Rightarrow \frac{d}{dt}\int_{\Omega} u \, dV = \int_{\Omega} \frac{\partial u}{\partial t} \, dV$$

2) Divergence theorem.

$$\int_{\partial\Omega} \underline{j} \cdot \hat{n} \, dA = \int_{\Omega} \nabla \cdot \underline{j} \, dV$$

substitute:

$$\int_\Omega \frac{\partial u}{\partial t}\, dV = -\int_\Omega \nabla \cdot \underline{j}\, dV + \int_\Omega \hat{f}_s\, dV$$

$$\int_\Omega \underbrace{\left(\frac{\partial u}{\partial t} + \nabla \cdot \underline{j} - \hat{f}_s\right)}_{g(x,t)} dV = 0$$

Why can we drop the integral?

$\Rightarrow$ localization

$$\int_\Omega g(x,t)\, dV = 0 \quad \Rightarrow \quad g(\underline{x},t) = 0 \;?$$

because Ω is arbitrary g has to be zero everywhere because otherwise we could choose Ω around the non-zero region.

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \nabla \cdot \underline{j} = \hat{f}_s}$$

Local form of general scalar balance law

$\rho_f = \frac{M_f}{\ldots}$ $\phi = \frac{V_f}{\ldots}$

Mass balance

Γ_f $\quad V_f$ $\quad u = \frac{M_f}{V_T}$ $\quad V_T$

ϕ = porosity

I Fluid mass balance

1) $u = \phi \rho_f$ "fluid mass per unit porous medium"

2) $\underline{j}(u) = u \underline{v}_f = \phi \rho_f \underline{v}_f = \rho_f q_f$ $\qquad q_f = \phi \underline{v}_f$

3) $\hat{f}_s = \rho_f \Gamma$ $\qquad \Gamma$ = is rate of melting

$\Rightarrow \quad \# = \left[\frac{M}{L^3}\right]$

Fluid mass balance:

$$\boxed{\frac{\partial}{\partial t}(\phi \rho_f) + \nabla \cdot (\rho_f q_f) = \rho_f \Gamma}$$

ϕv_f

II: Solid mass balance

1) $u = (1-\phi)\rho_s$

2) $\underline{j} = u v_s = \rho_s (1-\phi) \underline{v}_s$

3) $\hat{f}_s = -\rho_s \Gamma$

Solid mass balance

$$\boxed{\frac{\partial}{\partial t}\left((1-\phi)\rho_s\right) + \nabla \cdot \left(\rho_s (1-\phi) \underline{v}_s\right) = \rho_s \Gamma}$$

For now assume no phase change $\Gamma = 0$

if ρ_s = const & ρ_f = const $(\rho_s \neq \rho_f)$

then

I. $\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \underline{v}_f) = 0$

II $-\frac{\partial \phi}{\partial t} + \nabla \cdot ((1-\phi)\underline{v}_s) = 0$

Two phase continuity equation:

$\nabla \cdot [\phi \underline{v}_f + (1-\phi)\underline{v}_s] = 0$

$\nabla \cdot [\underbrace{\phi (v_f - v_s)}_{q_r \text{ Darcy's law}} + v_s] = 0$

$\boxed{\nabla \cdot [q_r + \underline{v}_s] = 0}$

q_r is given by Darcy

$\nabla \cdot \underline{v}_s = \dot{\varepsilon}_{vol}$ need constitutive law

1) Elastic rock

$$\nabla\cdot\underline{v}_s \equiv c_r \frac{\partial p_f}{\partial t}$$

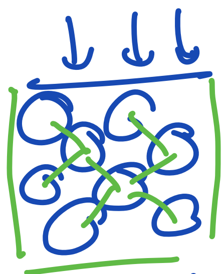
c_r = rock compressibility $\left[\frac{LT^2}{\mu}\right]$

$$c_r = -\frac{1}{V}\frac{dV_T}{d\sigma'}\Big|_T$$

$V_T = V_f + V_s$

σ' = effective stress

$\sigma_T = \sigma' + p_f$ = total stress



substitute into continuity

$$\nabla\cdot(\underline{q}_r + \underline{v}_s) = \nabla\cdot\underline{v}_s + \nabla\cdot\underline{q}_r$$
$$= c_r\frac{\partial p_f}{\partial t} - \nabla\cdot\left(\frac{k}{\mu}\nabla p_f + \rho_f g\hat{z}\right) = 0$$

Standard ground water equ

$$\boxed{c_r\frac{\partial p_f}{\partial t} - \nabla\cdot\left[\frac{k}{\mu}\left(\nabla p_f + \rho_f g\hat{z}\right)\right] = 0}$$

assumed σ_T = const

typically $c_r \approx 10^{-8}\frac{1}{Pa}$

implied porosity change $\phi \sim \phi_0 e^{c_r(p-p_0)}$

Δp due to 100 m water column

$\Delta p = \rho_s g h \sim 10^6\, Pa \quad \Rightarrow \quad \Delta\phi \sim \phi_0 e^{10^{-2}} \approx \phi_0$

$\Rightarrow$ porosity change is <u>negligible</u>

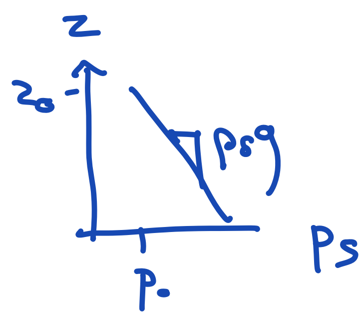
2) Ductile / viscous rock

Change the const. for $\dot{\varepsilon}_{vol}$

$p_f - p_s = \zeta \nabla \cdot \underline{v}_s \qquad \zeta$ = bulk/compaction viscosity

$p = \zeta \nabla \cdot v_s$

$p = p_f - p_s$ overpressure in fluid



Assume: $p_s = p_0 + \rho_s g (z_0 - z)$

Reformulate Darcy's law

$$q_r = -\frac{k}{\mu}\left(\nabla p_f + \rho_f g \hat{z}\right) = -\frac{k}{\mu}\left(\underbrace{\nabla p_f - \nabla p_s}_{\nabla p} + \underbrace{\nabla p_s}_{-\rho_s g \hat{z}} + \rho_f g \hat{z}\right)$$

$$\boxed{q_r = -\frac{k}{\mu}\left(\nabla p + \Delta\rho g \hat{z}\right)} \qquad \Delta\rho = \rho_f - \rho_s$$

Substitute into continuity

$$\nabla\cdot(q_r + \underline{v}_s) = \nabla\cdot q_r + \nabla\cdot v_s$$

$$= -\nabla\cdot\left[\frac{k}{\mu}(\nabla p + \Delta\rho g\hat{z})\right] + \frac{p}{\xi} = 0$$

Compaction equ: $\boxed{-\nabla\cdot\left[\frac{k}{\mu}(\nabla p + \Delta\rho g\hat{z})\right] + \frac{p}{\xi} = 0}$

$\Rightarrow$ mod. Helmholtz equ

Needs to be coupled to porosity evolution

Solid mass balance

$$-\frac{\partial\phi}{\partial t} + \nabla\cdot\left[(1-\phi)\underline{v}_s\right] = 0$$

$$-\frac{\partial\phi}{\partial t} - \nabla\cdot(\phi\underline{v}_s) + \underbrace{\nabla\cdot v_s}_{\frac{p}{\xi}} = 0$$

$$\Rightarrow \boxed{\frac{\partial\phi}{\partial t} + \nabla\cdot[\underline{v}_s\phi] = \frac{p}{\xi}}$$

III Rigid rock

$\underline{v}_s = 0 \qquad \underline{q}_r = \phi(v_f - \cancel{v_s}^{\,0}) = \phi v_f = \underline{q}_f$

continuity: $\nabla \cdot \underline{q}_r = 0$

substitute Darcy

$$-\nabla \cdot \left[\frac{k}{\mu} \left(\nabla p_f + \rho_f g \hat{z} \right) \right] = 0$$

simplify by introducing the head

$$h = z + \frac{p_f - p_0}{\rho_f g} \qquad [L]$$

$$p_f = p_0 + \rho_f g \, (h - z)$$

$$\nabla p_f = \rho_f g \, (\nabla h - \underbrace{\nabla z}_{\hat{z}})$$

sub. into Darcy

$$\underline{q}_r = -\frac{k}{\mu} \left(\rho_f g \, (\nabla h - \cancel{\hat{z}}) + \cancel{\rho_f g \hat{z}} \right)$$

$$\underline{q}_r = -\frac{k \rho_f g}{\mu} \nabla h = -K \nabla h$$

hydr. conductivity $\quad K = \frac{k \rho_f g}{\mu}$

$$\boxed{-\nabla \cdot [K \nabla h] = 0}$$ Laplace equ

$\hat{f}_s$

Incompressible flow in ridgid rock

Elastic rock (reform. in head)

$$S_r \frac{\partial h}{\partial t} - \nabla \cdot [K \nabla h] = 0$$

$\uparrow$

time dep. $\quad$ Diffusion equ

$$S_r = \rho_f g c_r$$