Lecture 3: Intro to numerics

Logistics: • Office hours Mon 4-5 pm

Wed 4-5 pm

• Check access to Matlab grader

• Next week in person

→ please fill out poll

Last time: Balance laws & model equations

Gen. balance law: $\boxed{\frac{\partial u}{\partial t} + \nabla \cdot \underline{j}(u) = \hat{f}_s}$

Mass balance:

fluid: $\frac{\partial \phi}{\partial t} + \nabla \cdot [\phi \underline{v}_f] = 0$

solid: $-\frac{\partial \phi}{\partial t} + \nabla \cdot [(1-\phi)\underline{v}_s] = 0$

(incompressible)

$\nabla \cdot \underline{v} = 0$

Two-phase continuity: $\boxed{\nabla \cdot [\underline{q}_r + \underline{v}_s] = 0}$

Rigid rock: $\underline{v}_s = 0 \Rightarrow \boxed{-\nabla \cdot [K \nabla h] = 0}$

$q_r = -\frac{k}{\mu}(\nabla p_f + \rho_f g \hat{z})$

$= -K \nabla h$

⇒ Laplace / Poisson Equ (elliptic)

• linear
• steady

} simplest

Elastic rock: $\nabla \cdot \underline{v}_s = S \frac{\partial h}{\partial t} = \cancel{c_r \frac{\partial p_f}{\partial t}} \Rightarrow S = c_r \rho_f g$

$$\boxed{S \frac{\partial h}{\partial t} - \nabla \cdot [K \nabla h] = 0}$$

$\Rightarrow$ Diffusion equation (parabolic)

- linear
- transient

neglect porosity change

Ductile rock: $\nabla \cdot \underline{v}_s = p_f - p_p \equiv p$

$$\boxed{\begin{aligned} &-\nabla \cdot \left[\frac{k}{\mu}(\nabla p + \Delta\rho g \hat{z})\right] + \frac{p}{\xi} = 0 \\ &-\nabla^2 \varphi = \frac{p}{\xi} \qquad v_s = -\nabla \varphi \\ &\frac{\partial \phi}{\partial t} + \nabla \cdot [\underline{v}_s \phi] = \frac{p}{\xi} \end{aligned}}$$

Non-linear system of equations.

we'll get back to this!

Today: - Review of Finite differences

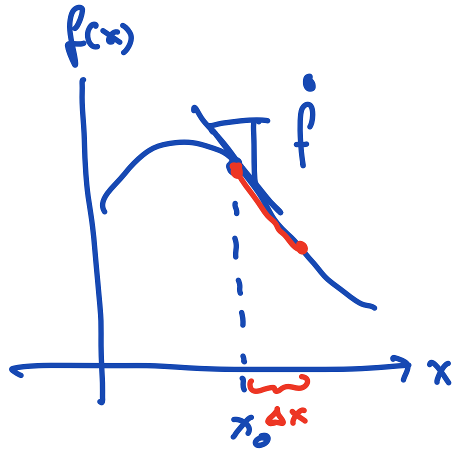
- Discrete operators

- Conservative finite differences

Finite differences

In calculus we define derivative

$$\dot{f} = \frac{df}{dx}\Big|_{x_0} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

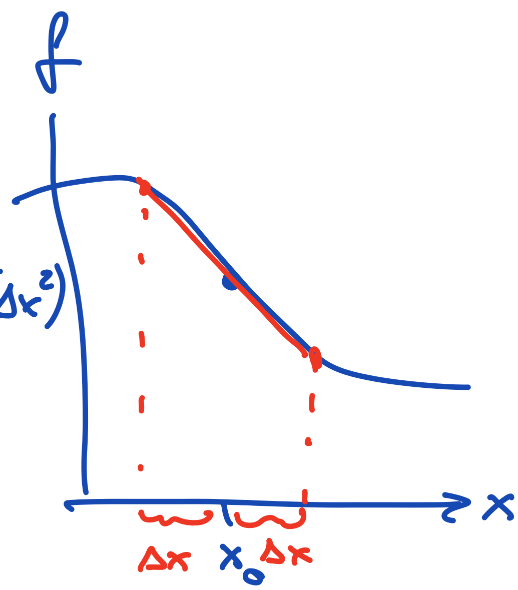


In finite diff. approximation

$$\hat{\dot{f}}(x) = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} + \mathcal{O}(\Delta x)$$

→ in numerical methods class you can show that this one-sided approx. is first order accurate

Central difference approx.



$$\hat{\dot{f}}(x_0) = \frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

⇒ our goto approx.

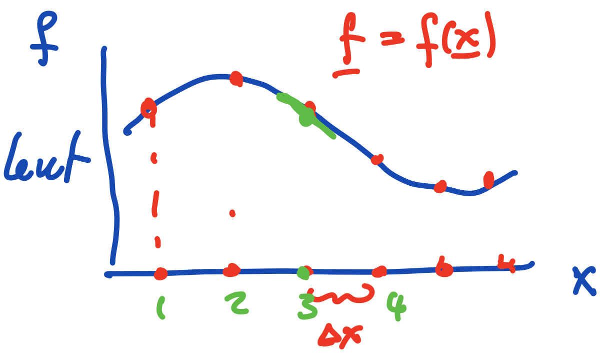
Differentiation matrix

Linear differential operator takes a function and returns another function

$$\dot{f}(x) = \mathcal{D}(f(x))$$

The discrete equivalent of function $\underline{f} = f(\underline{x})$

similarly $\underline{df} = \dot{f}(\underline{x})$

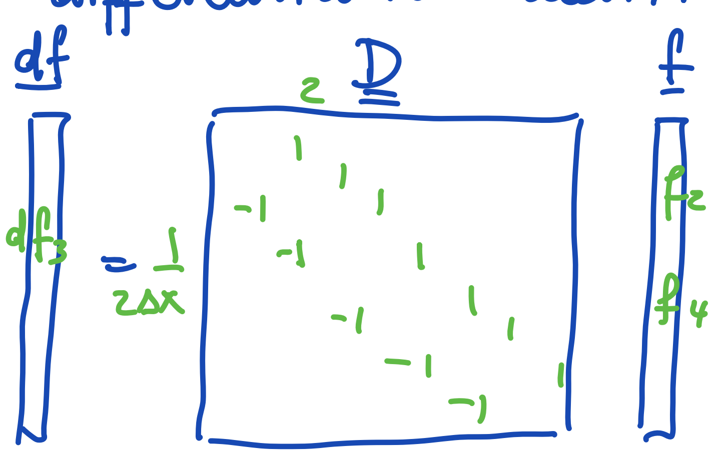


What is discrete equivalent of $\mathcal{D}$?

$$\underline{df} = \boxed{\underline{\underline{D}}}\ \underline{f}$$

has to be a matrix because it is linear and relates two vectors

⇒ differentiation matrix



$$df_3 = \frac{f_4 - f_2}{2\Delta x}$$

$\Rightarrow$ $\underline{\underline{D}}$ has simple bidiagonal structure

(on bnd we have to do something else)

What about 2nd derivative?

$$\underset{\partial^2 f/\partial x^2}{\underline{ddf}} = \underline{\underline{D}}\;\underline{df} = \underline{\underline{D}}\,\underline{\underline{D}}\,\underline{f}$$

Conservative finite differences

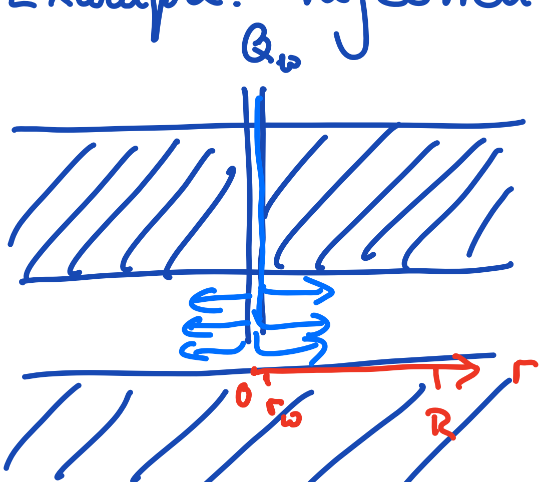
We would like to solve incompressible flow in a rigid rock.

$$\boxed{-\underline{\nabla}\cdot[K\underline{\nabla} h] = f_s}$$

is ~~co~~ indep. of coordinate system

subject to suitable BC.

Example: Injection well



naturally in cyl. coord.

$\nabla\cdot\underline{q} = \frac{1}{r}\frac{d}{dr}(r q_r) \qquad \underline{q} = q_r$

$\nabla h = \frac{dh}{dr}$

PDE: $-\frac{1}{r}\frac{d}{dr}\left(r\frac{dh}{dr}\right) = 0 \quad r\in[r_w, R]$

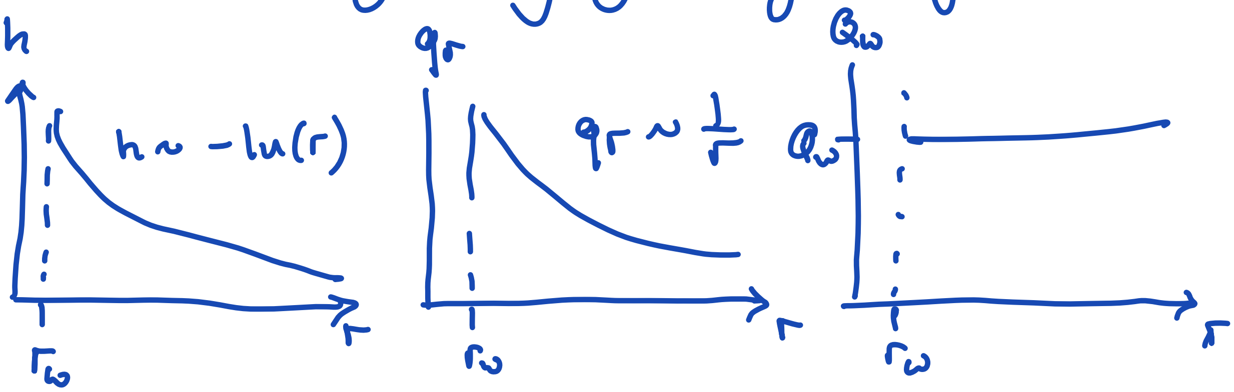
BC: $Q_w = A_w q_r(r_w) = -A_w K \frac{dh}{dr}\Big|$

$$\left.\frac{dh}{dr}\right|_{r_w}$$

$$\Rightarrow \left.\frac{dh}{dr}\right|_{r_w} = -\frac{Q_w}{A_w K} \quad \text{(Neumann BC)}$$

$$h(r=R) = h_B$$

$\Rightarrow$ solve analytically by integrating twice



Finite difference discretization

PDE: $\frac{d}{dr}\left(r \frac{dh}{dr}\right) = 0$

$$= r\frac{d^2h}{dr^2} + \frac{dh}{dr} = 0$$

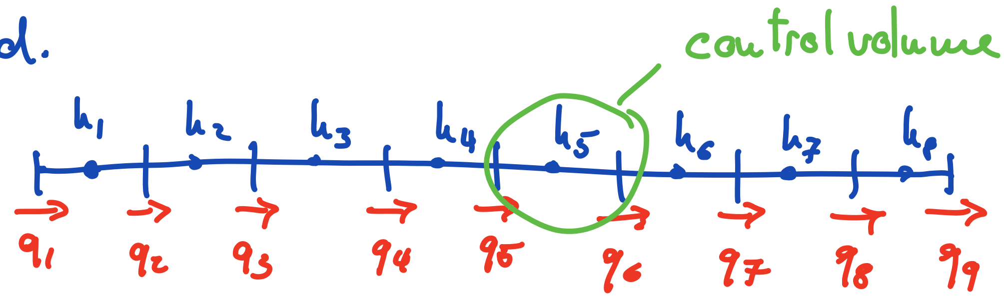
$$\frac{dh}{dr} = \underline{\underline{D}}\, \underline{h} = \frac{h_{i+1} - h_{i-1}}{2\Delta r}$$

$$\frac{d^2h}{dr^2} = \underline{\underline{D}}^2\, \underline{h} = \frac{h_{i+1} - 2h_i + h_{i-1}}{\Delta r^2}$$

$\rightarrow$ Live script

Conservative finite differences on staggered grid

To reduce width of FD stencil and to couple odd & even nodes we introduce a staggered grid.



divide grid into cells

approx unknown scalar in cell center

approx fluxes at cell faces

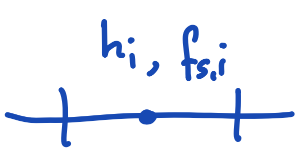
PDE: $$-\nabla\cdot[K\nabla h] = f_s$$

$$\nabla\cdot q = f_s \qquad \text{(mass balance)}$$

$$q = -K\nabla h \qquad \text{(const. law)}$$

Discretize div-grad system:

1) $$\nabla\cdot q = f_s \xrightarrow{1D} \frac{dq}{dx} = f \xrightarrow{FD} \frac{q_{i+1}-q_i}{\Delta x} = f_{s,i}$$



$\vec{q_i}$ $\vec{q_{i+1}}$

this central difference relative to location of f_i

2) $\underline{q} = -K\nabla h \rightarrow q_i = -K \dfrac{h_i - h_{i-1}}{\Delta x}$

assume for moment $K = $ const.

substitute 2 into 1.

$$-\frac{K}{\Delta x}\Big[\underbrace{\frac{h_{i+1}-h_i}{\Delta x}}_{q_{i+1}} - \underbrace{\frac{h_i - h_{i-1}}{\Delta x}}_{q_i}\Big] = f_{s,i}$$

$$-\frac{K}{\Delta x^2}\left[h_{i+1} - 2h_i + h_{i-1}\right] = f_{s,i}$$

now we have tight stencil $\rightarrow$ no decoupling

this is the std second order second deriv. stencil