Lecture 5: Shallow Aquifer Models & Boundary Conditions

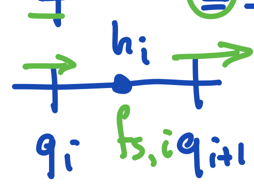
Logistics:
- Today online (Marc is sick)
- Thursday will be freezing
  - → likely UT will open late
  - → stay online for now
- HW 1 is due Th 9:30 am
- Office hours 4-5 pm Wed.

Last time: - Discrete operators

1) $\nabla \cdot \underline{q} = \underline{f_s} \rightarrow \underline{\underline{D}}\, \underline{q} = \underline{f_s}$

2) $\underline{q} = -K \underline{\nabla h} \rightarrow \underline{q} = -K \underline{\underline{G}}\, \underline{h}$

Staggered grid



$$\underline{\underline{G}} = \frac{1}{\Delta x}\begin{bmatrix} 0 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ & & & & 0 \end{bmatrix}$$

N_x+1 by N_x

cell centers → cell faces

$$\underline{\underline{D}} = \frac{1}{\Delta x}\begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \end{bmatrix}$$

N_x by N_x+1

faces → centers

Today: • Introduce shallow aquifer model

→ 1D model problem

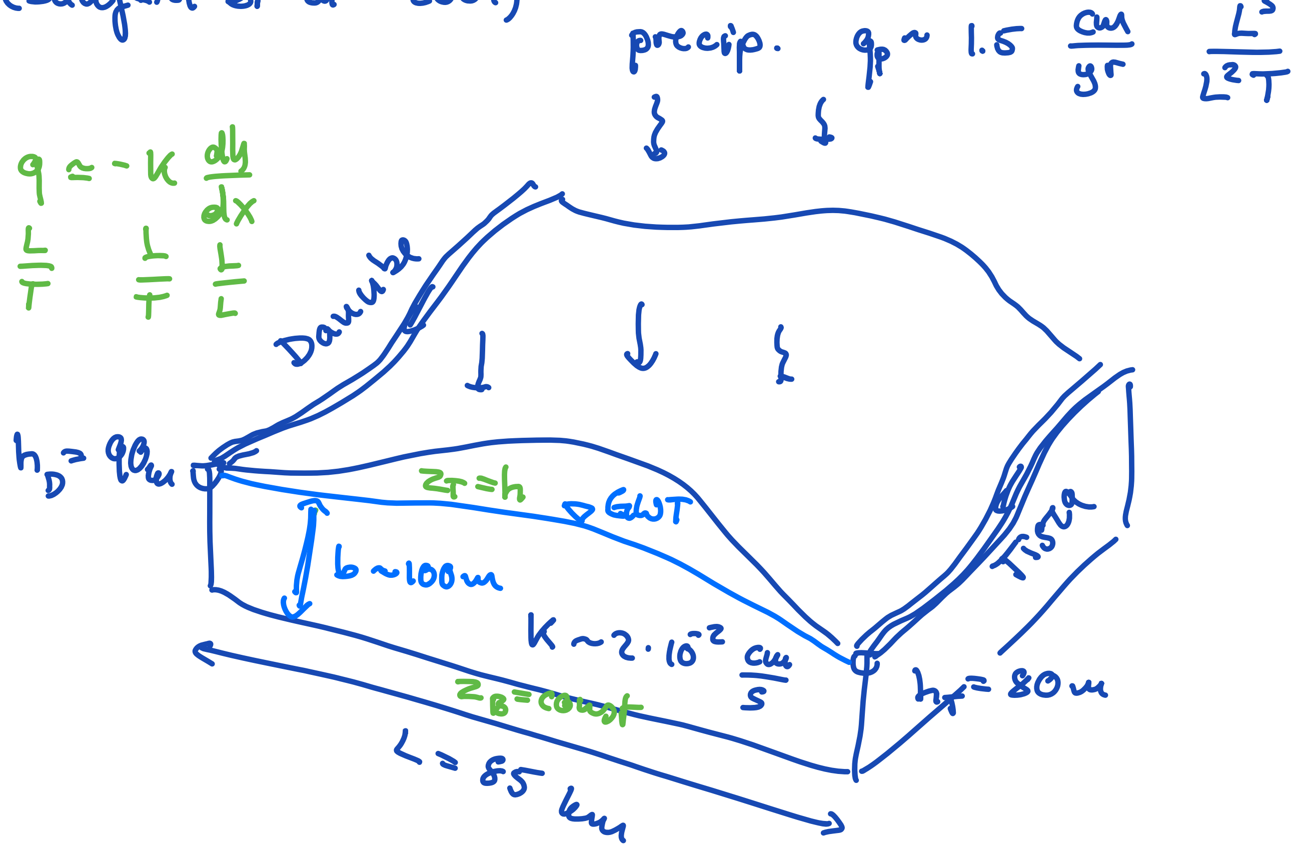
→ motivate BC's

• Dirichlet BC's

→ using constraints

Groundwater recharge between two rivers

(Sanford et al 2001)



Aquifer aspect ratio: $b/L = \frac{100\,m}{\text{[illegible]}} \approx \frac{1}{\text{[illegible]}}$

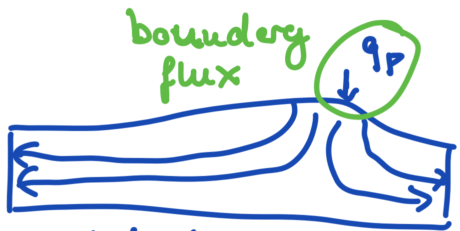
85 000 m 1000

$\Rightarrow$ flow is practically 1D in horizontal direction

This can be seen from scaling analysis of the continuity eqn.:

$x_D = \frac{x}{L}$ $\qquad z_D = \frac{z}{b}$ $\qquad q_{x,D} = \frac{q_x}{q_{x,c}}$ $\qquad q_{z,D} = \frac{q_z}{q_{z,c}}$

$q_{z,c} = q_p$



$$q_x = \underbrace{q_{x,c}}_{\text{const}} \underbrace{q_{x,D}}_{\text{varies}}$$

substitute into continuity eqn:

$$\nabla \cdot q = \frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z} = \frac{q_{x,c}}{L} \frac{\partial q_{x,D}}{\partial x_D} + \frac{q_{z,c}}{b} \frac{\partial q_{z,D}}{\partial z_D} = 0$$

collect terms

$$\frac{\partial q_{x,D}}{\partial x_D} + \underbrace{\frac{q_{z,c}\, L}{q_{x,c}\, b}}_{\Pi} \frac{\partial q_{z,D}}{\partial z_D} = 0$$

$\Pi = 1$ suggest the relation between scales

$$\frac{q_{z,c}}{q_{x,c}} \frac{L}{b} = 1 \quad \Rightarrow \quad q_{z,c} = \frac{b}{L} q_{x,c} \sim 10^{-3} q_{x,c}$$

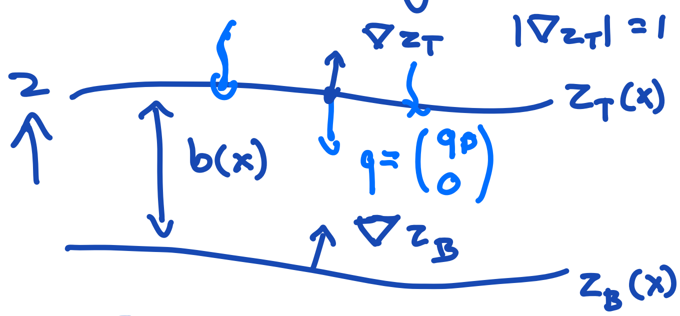
the vertical fluxes are approx. 1000 times smaller than horizontal flux!

Assume $q_z = 0 \Rightarrow \frac{\partial h}{\partial z} = 0 \Rightarrow h = h(x)$

implies that pressure is hydrostatic in vertical.

Darcy: $q_x = -K \frac{\partial h}{\partial x}$

Vertical integration



$b(x) = z_T(x) - z_B(x)$

$\nabla \cdot q = 0$

$$\int_{z_B}^{z_T} \nabla \cdot q \, dz \overset{?}{=} \nabla \cdot \int_{z_B}^{z_T} q \, dz$$

Leibnitz integral rule:

$$\int_{z_B}^{z_T} \nabla \cdot q \, dz = \nabla \cdot \int_{z_B}^{z_T} q \, dz + \left(q \cdot \nabla z_B \right|_{z_B} - \left(q \cdot \nabla z_T \right|_{z_T}$$

Assumptions:

1) $q = \begin{pmatrix} q_x(x) \\ 0 \end{pmatrix} \Rightarrow q \neq q(z)$

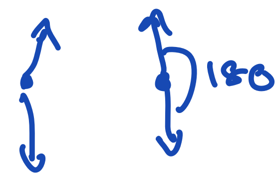
$$\int_{z_B}^{z_T} \underline{q}\, dz = \underline{q}\,(z_T - z_B) = b(x)\,\underline{q}(x)$$

2) bottom of aquifer is impermeable

$$\underline{q}\cdot\nabla z_B\big|_{z_B} = \underline{q}\cdot\hat{n}_B\big|_{z_B} = 0$$

3) slope of top of aquifer is negligible

$$\underline{q}\cdot\nabla z_T\big|_{z_T} = \underline{q}\cdot\hat{n}_T\big|_{z_T} = -q_p$$



Substitute into vertically integrated eqn.

$$\int_{z_B}^{z_T} \nabla\cdot q\, dz = 0 \quad\Rightarrow\quad -\tilde{\nabla}\cdot[b K \tilde{\nabla} h] = q_p$$

$\tilde{\nabla}$ horizontal components of operator

full eqn: $\quad -\nabla\cdot[K\nabla h] = 0 \qquad \underline{q}\cdot\hat{n}_T\big|_{z_T} = q_p$

3D or 2D $\qquad\qquad\qquad\qquad \underline{q}\cdot\hat{n}_B\big|_{z_B} = 0$

shallow approx: $-\tilde{\nabla}\cdot[b\, K\, \tilde{\nabla} h] = q_p$

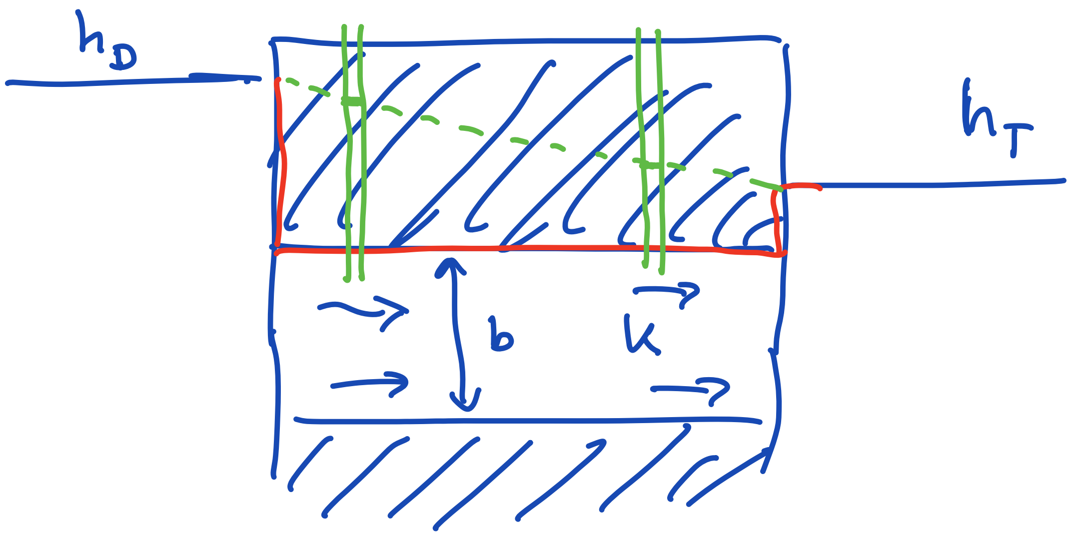
In 1D $\qquad -\frac{d}{dx}\left[b K \frac{dh}{dx}\right] = q_p$

Note:

In an unconfined aquifer $z_T = h$ free surface

$$-\tilde{\nabla} \cdot [K\, h\, \tilde{\nabla} h] = q_p \qquad \text{non-linear}$$

Here we assume $b = \text{const} \Rightarrow$ confined aquifer



linear problem → start.

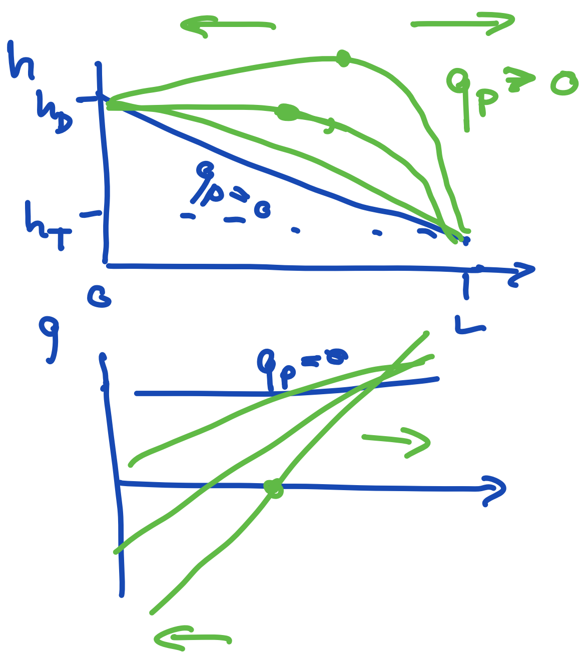
Simplified example problem:

PDE: $\quad -\frac{d}{dx}\left[b\, K \frac{dh}{dx}\right] = q_p \qquad x \in [0, L]$

BC: $\quad h(0) = h_D \qquad h(L) = h_T$

Integrate twice to obtain analytic solution

$$\boxed{\begin{aligned} h &= h_D + \left(\frac{h_T - h_D}{L} + \cancel{\frac{q_p L}{2bK}}\right) x - \frac{q_p}{2bK}\, x^2 \\ q &= \frac{\cancel{q_p} \to 0}{b}\left(x - \frac{L}{2}\right) - \frac{K}{L}\left(h_T - h_D\right) \end{aligned}}$$



groundwater divide

Dirichlet boundary conditions

BC's are necessary for the problem to be well posed, i.e., unique solution. Dirichlet BC's prescribe the unknown on bnd. This provides constraints on ~~soluti~~ unknown that reduce the number of unknowns we need to solve for.

$\Rightarrow$ need to understand how to eliminate constraints from our system.

Example 1: Homogeneous BC's

PDE: $-\frac{d}{dx}\left[bk\frac{dh}{dx}\right] = q_r \qquad x \in [0, L]$

BC: $h(0) = h(L) = 0$

We can use discrete operators to write PDE as system of linear equations

$$\underline{\underline{L}}\,\underline{h} = \underline{f_s} \qquad \underline{\underline{L}} = -\underline{\underline{D}} * \underline{\underline{G}} \quad \underline{f_s} = \frac{q_r}{bk}\underline{1}$$

$\underline{\underline{L}}$ is not invertible!

$\Rightarrow$ need to eliminate constraints from BC to make invertible matrix

Write ~~bc~~ BC's as a linear system.

$$h(0) = 0 \qquad h(L) = 0$$

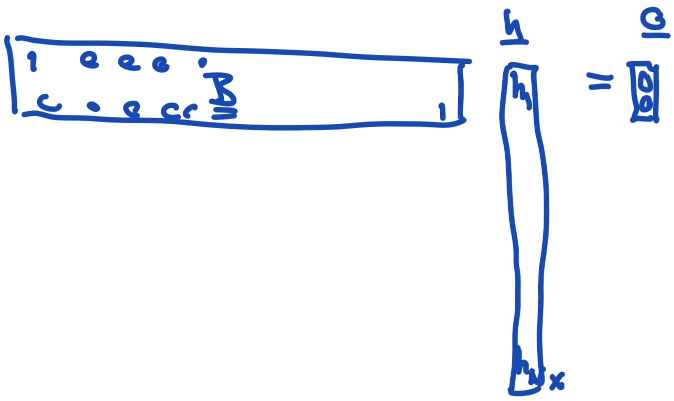
$$h_1 = 0 \qquad h_{N_x} = 0$$

Note: Imposing Dir. BC's at cell center.

$$\underline{\underline{B}}\,\underline{h} = 0$$

$\underline{\underline{B}}$ is a N_c by N_x constraint matrix

↑ # of constraints



$h_1 = 0$

$0 = 0$

Full discrete problem statement

PDE : $\boxed{\underline{\underline{L}}\,\underline{h} = \underline{f_s}}$ — $\underline{\underline{L}}$ is N_x by N_x system matrix

BC : $\boxed{\underline{\underline{B}}\,\underline{h} = \underline{0}}$ — $\underline{\underline{B}}$ is N_c by N_x constraint mat.

Neither system is solvable individually

$\Rightarrow$ need to combine them by eliminating the constraints $\underline{\underline{B}}$ from $\underline{\underline{L}}$ to produce a "reduced" system $\underline{\underline{L_r}}$ $(N_x - N_c)$ by $(N_x - N_c)$