Lecture 6: Dirichlet BC's

Logistics : - Next class in person/hybrid

- HW1 is due
- HW2 depends how far we get today

Last time: - shallow aquifer model

PDE: $-\nabla \cdot [\underline{bk} \nabla h] = \underline{q_p}$ $\quad x \in [0, L]$

BC: $h(0) = h_D \qquad h(L) = h_T$

- Discrete problem

$$\boxed{\begin{aligned} \underline{\underline{L}}\, \underline{h} &= \underline{f_s} \\ \underline{\underline{B}}\, \underline{h} &= \underline{g} \end{aligned}}$$

$\underline{\underline{L}} = -bK\, \underline{\underline{D}}\, \underline{\underline{G}}$

$N_c \times N_x$ $\quad \underline{g} = \begin{bmatrix} h_D \\ h_T \end{bmatrix}$

Today: - Elimination of the constraints

- solution to reduced system
- homogeneous case $h(0) = h(L) = 0$
- heterogeneous case

Reduced linear system

Constraints remove dof's

$\Rightarrow$ expect to solve a smaller linear system

for the remaining ~~an~~ dof's

Reduced system: $\boxed{\underline{\underline{L}}_r \, \underline{h}_r = \underline{f}_{s,r}}$

if N_c is # of constraints

$\underline{h}_r$ is $(N_x - N_c) \times 1$ red. solu. vector

$\underline{f}_{s,r}$ is $(N_x - N_c) \times 1$ red. rhs vector

$\underline{\underline{L}}_r$ is $(N_x - N_c) \times (N_x - N_c)$ red. sys. matrix

Projection matrix

What is the relation between $\underline{h}_r$ and $\underline{h}$?

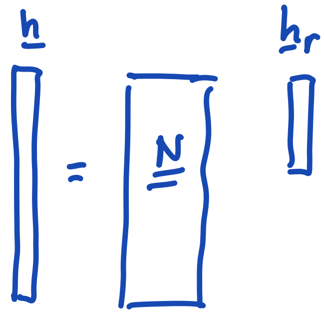
$\underline{f}_{s,r}$ and $\underline{f}_s$?

$\underline{\underline{L}}_r$ and $\underline{\underline{L}}$?

Remember everything is linear!

$\Rightarrow$ two vectors of different length are related by a rectangular matrix.

$$\underset{N_x \cdot 1}{\underline{h}} = \underset{N_x \cdot (N_x - N_c)}{\underline{\underline{N}}} \qquad \underset{(N_x - N_c)\cdot 1}{\underline{h}_r}$$



What is $\underline{\underline{N}}$?

For now we just require $\underline{\underline{N}}$ is <u>orthonormal</u>

If $\underline{n}_i$ is the i-th column of $\underline{\underline{N}} = \begin{bmatrix} | & | & | & \\ \underline{n}_1 & \underline{n}_2 & \underline{n}_3 & \dots \\ | & | & | & \end{bmatrix}$

then $\underline{n}_i^T \cdot \underline{n}_i = 1$

$\underline{n}_i^T \cdot \underline{n}_j = 0 \qquad i \neq j$

It follows that

a) $\underset{(N_x - N_c)\cdot \cancel{N_x}}{\underline{\underline{N}}^T} \; \underset{\cancel{N_x}\cdot (N_x - N_c)}{\underline{\underline{N}}} = \underset{(N_x - N_c)\cdot(N_x - N_c)}{\underline{\underline{I}}_r}$ identity in reduced space

b) $\underset{N_x \cdot \cancel{(N_x - N_c)}}{\underline{\underline{N}}} \; \underset{\cancel{(N_x - N_c)}\cdot N_x}{\underline{\underline{N}}^T} \qquad \underset{N_x \cdot N_x}{\underline{\underline{I}}'}$ "identity" matrix but it has N_c zeros on diagonal

If this is the case and $\underline{h} = \underline{\underline{N}}\, \underline{h}_r$

$$\underline{\underline{N}}^T \underline{h} = \underbrace{\underline{\underline{N}}^T \underline{\underline{N}}}_{I_r} \underline{h}_r$$

So that $\boxed{\begin{array}{l} \underline{h} = \underline{\underline{N}}\, \underline{h}_r \\ \underline{h}_r = \underline{\underline{N}}^T \underline{h} \end{array}}$ $\underline{\underline{N}}$ is a matrix that allows us to go forth and back between full & reduced solution space

We say $\underline{\underline{N}}^T$ projects the vector of unknowns into the reduced solution space.

(Note: In lin. algebra a projection matrix is square)

Similarly $\underline{f}_s = \underline{\underline{N}}\, \underline{f}_{sr}$ $\underline{f}_{sr} = \underline{\underline{N}}^T \underline{f}_s$

How is the system matrix projected into the reduced space?

$$\underline{\underline{L}}\, \underline{h} = \underline{f}_s$$

$$\underline{\underline{N}}^T \underline{\underline{L}}\, \underline{h} = \underline{\underline{N}}^T \underline{f}_s = \underline{f}_{s,r} \qquad \text{sub. } \underline{h} = \underline{\underline{N}}\, \underline{h}_r$$

$$\underbrace{\underline{\underline{N}}^T \underline{\underline{L}}\, \underline{\underline{N}}}_{\underline{\underline{L}}_r}\, \underline{h}_r = \underline{f}_{s,r}$$

Reduced system matrix : $\boxed{\underline{\underline{K}}_r = \underline{\underline{N}}^T \underline{\underline{K}}\, \underline{\underline{N}}}$

Now we just need to find $\underline{\underline{N}}$!

$\underline{\underline{N}}$ needs to contain information about

the boundary conditions, $\underline{\underline{B}}$.

In particular the location of constraints

Null space of constraint matrix $\underline{\underline{B}}$

In which space should we look for a solution?

$\Rightarrow$ Any solution that satisfies the BC's

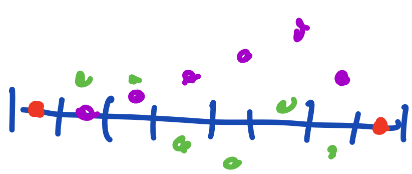
ie the constraints.

All $\underline{h}$ that satisfy $\underline{\underline{B}}\,\underline{h} = \underline{0}$ $\Rightarrow$ all vectors

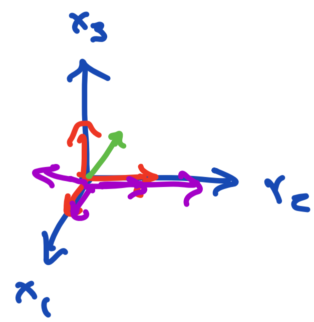
that are zero on boundary.

This is the null space $\mathcal{N}(\underline{\underline{B}})$ of the constraint

matrix.



The matrix $\underline{\underline{N}}$ can be any orthonormal basis for $\mathcal{N}(\underline{\underline{B}})$.



$\mathbb{R}^3$

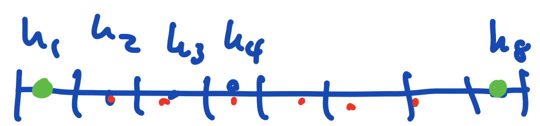
A basis is a collection of vectors that allow you to access any point in the vectorspace by linear combination.

In Matlab we can find a null space : $\underline{\underline{N}} = \text{null}(\underline{\underline{B}})$

(download) $\underline{\underline{N}} = \text{spnull}(\underline{\underline{B}})$

However, this is slow for large systems.

It turns out it is easy to find basis for $\mathcal{N}(\underline{\underline{B}})$



BC's set $h_1 = h_8 = 0$

$$\underline{\underline{I}}_{Nx \cdot Nx} = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

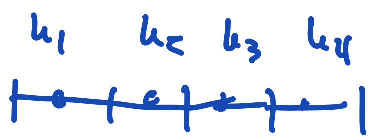
$$\underline{\underline{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{N}} = \begin{bmatrix} 0 & & & & & \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}$$

$\left[\quad \quad \quad \quad e \right]$

$\underline{\underline{N}}\,\underline{h}$

You can think of spliting the identity into $\underline{\underline{B}}$ (rows corresponding to Ne constraints) and $\underline{\underline{N}}$ (columns corresponding to Nx-Ne actual unknowns)



$$h_1 = h_4 = 0$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{N}} = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}$$

$$\underline{h} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \notin N(\underline{\underline{B}})$$

$$\underline{\underline{N}}\,\underline{h} = \underline{h}_r = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix} \in \mathcal{N}(\underline{\underline{B}})$$

Note on implementation:

```
dof_dir = [Grid.dof_xmin; Grid.dof_xmax]
B = I(dof_dir, :);
% Build N is by deleting these rows
```

$\underline{\underline{N}} = \underline{\underline{I}};$

$\underline{\underline{N}}$(:, dof_dsr) = [];

Step 1: create N

Step 2: create $\underline{\underline{L}}_r$ and $\underline{f}_{sr}$

Step 3: solve for $\underline{h}_r$

Step 4: $\underline{h} = \underline{\underline{N}}\, \underline{h}_r$

Heterogeneous Dirichlet constraints

$\Rightarrow \quad \underline{\underline{B}}\, \underline{h} = \underline{g} \quad$ where $\quad \underline{g} = \begin{bmatrix} h_D \\ h_T \end{bmatrix}$

Here we decompose $\underline{h} = \underline{h}_0 + \underline{h}_p$ into

homogeneous solution $\underline{h}_0$ and a particular soln $\underline{h}_p$

$$\left.\begin{aligned} &\text{homog:} \quad \underline{\underline{B}}\, \underline{h}_0 = \underline{0} \\ &\text{hetero:} \quad \underline{\underline{B}}\, \underline{h}_p = \underline{g} \end{aligned}\right\} \quad \underline{\underline{B}}\,(\underbrace{\underline{h}_0 + \underline{h}_p}_{\underline{h}}) = \underline{g}$$

Note: $\underline{h}$ is unique (assuming approp. BC)

Splitting into $\underline{h}_0$ & $\underline{h}_p$ is not unique
but there is simplest choice.

Two questions:

1) how do we determine suitable $\underline{h}_p$

2) Given $\underline{h}_p$ what is the associated $\underline{h}_0$?

Start with step 2: Suppose we have $\underline{h}_p$

$\underline{\underline{L}}(\underline{h}_0 + \underline{h}_p) = \underline{f}_S$ $\qquad$ h_p is known $\to$ rhs

$$\underline{\underline{L}}\,\underline{h}_0 = \underline{f}_S - \underbrace{\underline{\underline{L}}\,\underline{h}_p}_{f_D}$$

$\underline{\underline{L}}\,\underline{h}_0 = \underline{f}_S + \underline{f}_D$ $\qquad$ $f_D = -\underline{\underline{L}}\,\underline{h}_p$

To solve this we proceed as before

$$\underbrace{\underline{\underline{N}}^T \underline{\underline{L}}\,\underline{\underline{N}}}_{\underline{\underline{L}}_r}\,\underbrace{\underline{\underline{N}}^T \underline{h}_0}_{\underline{h}_{0r}} = \underline{\underline{N}}^T(\underline{f}_S + \underline{f}_D) = N^T \underline{f} \qquad \underline{f} = \underline{f}_S + \underline{f}_D$$

$$\underline{\underline{N}}\,\underline{\underline{N}}^T = \underline{\underline{I}}$$

$$= \underline{f}_r$$

$\underline{\underline{L}}_r \quad \underline{h}_{0r} = \underline{f}_r \qquad \Rightarrow \qquad \underline{h}_0 = \underline{\underline{N}}\,\underline{h}_{0r}$

Finding a particular solution h_p:

Note that h_p does not need to satisfy

$\underline{\underline{L}}\,\underline{h}_p = \underline{f}_s$ it only needs to satisfy $\underline{\underline{B}}\,\underline{h}_p = \underline{g}$

reduced particular soln:

$$\underline{h}_{pr} = \underline{\underline{B}}\,\underline{h}_p \Rightarrow \underline{\underline{B}}\,\underline{h}_p = \underline{g} \Rightarrow \boxed{\underline{h}_{pr} = \underline{g}}$$

To recover full $\underline{h}_p$ from $\underline{h}_{pr}$ we

$$\boxed{\underline{h}_p = \underline{\underline{B}}^T \underline{h}_{pr}}$$

Can solve $\boxed{\underbrace{\underline{\underline{B}}\,\underline{\underline{B}}^T}_{N_e \text{ by } N_e} \underline{h}_{pr} = \underline{g}}$

for our simple constraints $BB^T = \underset{N_e \cdot N_e}{I}$

Summary: Solving linear boundary value Problems

$\underline{\underline{L}}\,\underline{h} = \underline{f}_s$

$\underline{\underline{B}}\,\underline{h} = \underline{g}$

Step 1: Find particular solution

$$\underline{h}_p = \underline{\underline{B}}^T \underline{h}_{pr} \qquad \underline{h}_{pr} = (\underline{\underline{B}}\,\underline{\underline{B}}^T)^{-1} \underline{g}$$

$$\boxed{h_p = B^T (B B^T)^{-1} g}$$

Step 2: Find associated hom. solution

$$\underline{\underline{N}}^T \underline{\underline{L}}\,\underline{\underline{N}}\,\underline{\underline{N}}^T \underline{h}_{or} = \underline{\underline{N}}^T (\underline{f}_s - \underline{\underline{L}}\,\underline{h}_p)$$

$$\underline{h}_0 = \underline{\underline{N}}\,\underline{h}_{or}$$

$$\boxed{\underline{h}_0 = \underline{\underline{N}}\left[(\underline{\underline{N}}^T \underline{\underline{L}}\,\underline{\underline{N}})^{-1} \underline{\underline{N}}^T (\underline{f}_s - \underline{\underline{L}}\,\underline{h}_p)\right]}$$

Step 3: add

$$\underline{h} = \underline{h}_0 + \underline{h}_p$$

⇒ all this will be enclosed into function

$$\underline{h} = \text{solve\_lbvp}(\underline{\underline{L}}, \underline{f}_s, \underline{\underline{B}}, \underline{g}, \underline{\underline{N}})$$