Lecture 7: Heterogeneous Coefficients

Logistics: - HW2 due Th

- Office hrs 4-5pm → zoom

Last time: - Solving linear systems with constraints

⇒ implementing Dirichlet BC

Today: - Variable coefficients

$$-\nabla\cdot[\underset{\uparrow}{K}\nabla u] = f_s$$

$K = K(\underline{x})$

- Layered media

→ classic analysis informs numerical implementation.

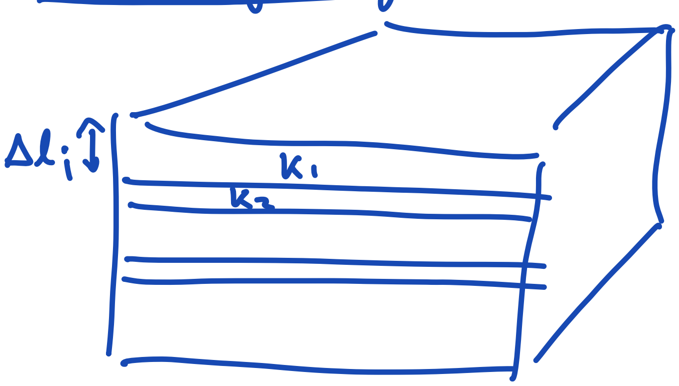
Layered Media

→ in geoscience "everything" is layered at many scales

Problem is we cannot represent all these layers explicitly even on powerful computers but even small scale layering has strong effect on solution

Main question is: Can we upsale this layering to capture its main effect on the flow?

Stack of layers



N layers

cond. k_i

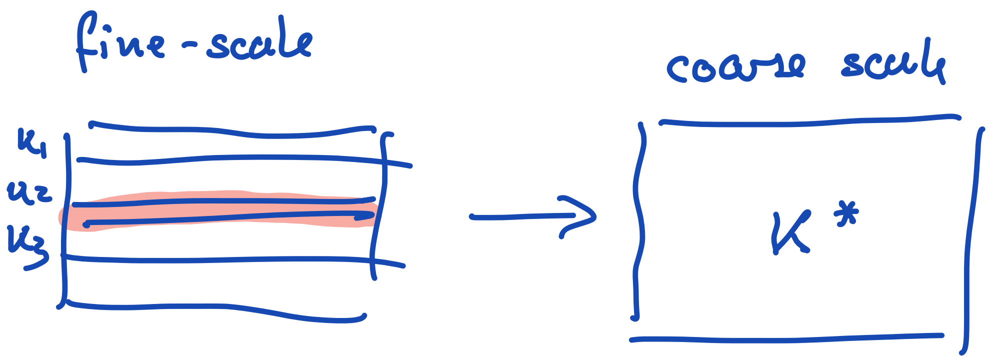
width Δl_i

$$L = \sum_{i=1}^{N} \Delta l_i$$

Two limiting cases

I) Flow along the layers

II) Flow across the layers



Would this effective K^* be same for flow along & across layers?

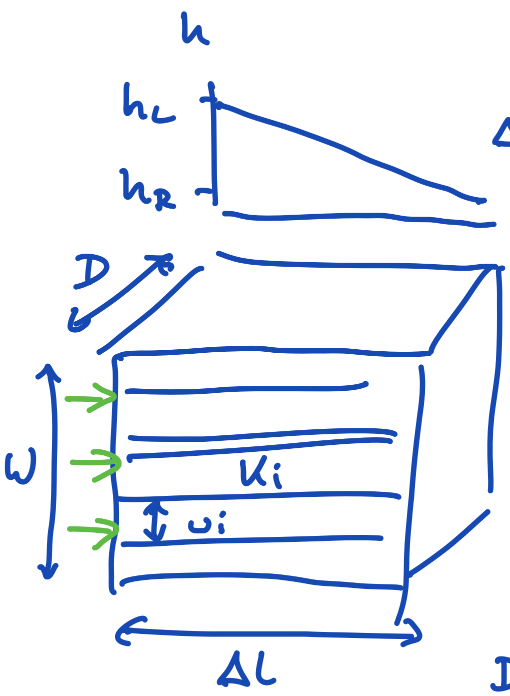
No $\rightarrow$ it will depend on direction of flow.

$\Rightarrow$ anisotropy

Fine scale: heterogeneous $\quad K = K(\underline{x})$

Coarse scale: anisotropic $\quad \underline{\underline{K}} = \begin{bmatrix} K^*_{\parallel} & \\ & K^*_{\perp} \end{bmatrix}$

I Flow along layers



$\Delta h = h_R - h_L$

flow is 1D along each layer

$\Rightarrow$ head is linear

Darcy in i-th layer:

$$Q_i = -D\, w_i\, K_i\, \frac{\Delta h}{\Delta L}$$

Darcy for whole stack:

$$Q = -D W K^*_{\parallel} \frac{\Delta h}{\Delta L}$$

$K^*_{\parallel}$ is effective hydraulic cond. for flow along layers.

Aim: $K^*_{\parallel} = K^*_{\parallel}(K_1, K_2, K_3 \ldots)$

Total flow rate: $Q = \sum_{i=1}^{N} Q_i$

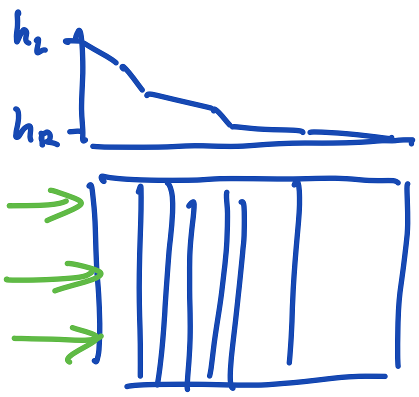
$$\cancel{-}\cancel{D} W K^*_{\parallel} \cancel{\frac{\Delta h}{\Delta L}} = \sum_{i=1}^{N} \left(-D\, w_i\, K_i\, \frac{\Delta h}{\Delta L}\right)$$

$$= \cancel{-} \cancel{D \frac{\Delta h}{\Delta L}} \sum_{i=1}^{N} w_i k_i$$

$$W K^*_{\parallel} = \sum_{i=1}^{N} w_i k_i$$

$\Rightarrow \boxed{K^*_{\parallel} = \sum_{i=1}^{N} \frac{w_i}{W} k_i}$ weighted arithmetic mean

II Flow across layers



all layers experience the same $q = -K_i \frac{\Delta h_i}{\Delta l_i}$

$\Rightarrow$ piecewise linear head profile

Darcy in each layer: $q = -K_i \frac{\Delta h_i}{\Delta l_i}$

Darcy across entire stack: $q = -K_\perp^* \frac{\Delta h}{\Delta L}$

$$\Delta h = \sum_{i=1}^{N} \Delta h_i \qquad \Delta L = \sum_{i=1}^{N} \Delta l_i$$

$$\Delta h_i = -\frac{q}{K_i} \Delta l_i$$

$$\Delta h = \sum_{i=1}^{N} -q \frac{\Delta l_i}{K_i}$$

$$K_\perp^* = -\frac{q \, \Delta L}{\Delta h} = \frac{\cancel{q} \, \Delta L}{\sum_{i=1}^{N} \cancel{q} \frac{\Delta l_i}{K_i}} = \frac{\Delta L}{\sum_{i=1}^{N} \frac{\Delta l_i}{K_i}} = \frac{1}{\sum_{i=1}^{N} \frac{\Delta l_i / \Delta L}{K_i}}$$

$$\boxed{K_\perp^* = \frac{1}{\sum_{i=1}^{N} \frac{\Delta l_i / \Delta l}{K_i}}}$$

harmonic average

weighted