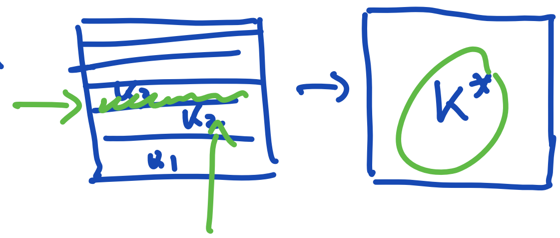
Lecture 8: Heterogeneous Media & fluxes

Logistics: - HW2 is due

- HW3 will be posted → comp_mean

Last time: - Review of Dirichlet BC's

- Layered media



Parallel flow: $K_{\parallel}^* = \sum_{i=1}^{N} \frac{w_i}{W} K_i$ (arithmetic)

Perpendicular flow : $K_{\perp}^* = \frac{1}{\sum_{i=1}^{N} \frac{\Delta l_i / \Delta l}{K_i}}$ (harmonic)

⇒ anisotropy $\underline{\underline{K}}^*$ is a tensor

Today: - Discretization of heterogeneous coefficients

- ~~Fluxes and Neumann~~ BC's

- Radial & spherical ops

Discretization of heterogeneous coefficients

Heterogenity is key element of geologic media

Continuous equ: $-\nabla \cdot [K(\underset{\sim}{x}) \nabla h] = f_s$

Discrete equ: $-\underline{\underline{D}} \cdot [\underline{\underline{Kd}} \; \underline{\underline{G}}] \underline{h} = \underline{f_s}$

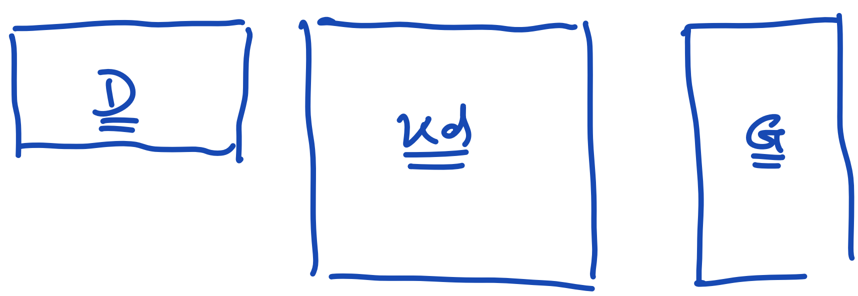
What is size of Kd? $N = \#$ cells N_f # faces

D	Kd	G
N by Nf	Nf by Nf	Nf by N

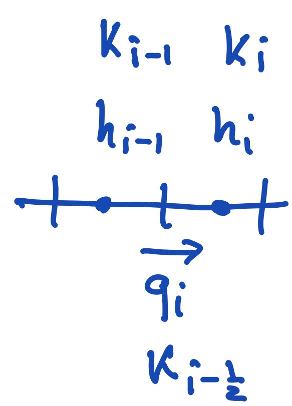
$N_f = N_x + 1$

$N = N_x$

$\Rightarrow$ $\underline{\underline{Kd}}$ if Nf by Nf matrix associated with faces



Entries into $\underline{\underline{Kd}}$ matrix



Darcy flux $\underline{q} = -K \nabla h$

$\underline{q} = -\underline{\underline{Kd}} \; \underline{\underline{G}} \; \underline{h}$

$q_i = -K_{i-\frac{1}{2}} \dfrac{h_i - h_{i-1}}{\Delta x}$

K_i's are typically defined on cells

for example K is often related to porosity which is naturally ~~evaluated~~ associated with cells

$\Rightarrow$ average K_i to faces

$\underline{\text{Kmean}}$ is N_f by 1 vector is $K_{i-1/2}$

$$q = -\underline{\text{Kmean}} \;.*\; (\underbrace{\underline{\underline{G}}\,\underline{h}}_{\underline{dh}}) \qquad \text{element wise mult.}$$

y

But to form $\underline{\underline{L}} = -\underline{\underline{D}}\;\underline{\text{Kmean}}\;\underline{\underline{G}}$

instead of vector we form a diagonal matrix

$$\underline{\text{Kmean}} \;.*\; \underline{dh} = \underline{\underline{Kd}} * \underline{dh}$$

$$\Rightarrow \quad \underline{\underline{L}} = -\underline{\underline{D}} * \underline{\underline{Kd}} * \underline{\underline{G}}$$

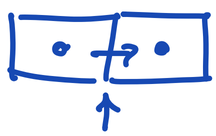
$\underline{\underline{Kd}}$ is ~~diagonal~~ N_f by N_f matrix with $\underline{\text{Kmean}}$ on diagonal

$$\underline{\underline{Kd}} = \begin{bmatrix} \ddots & & \\ & \text{Kmean} & \\ & & \ddots \end{bmatrix}$$

The appropriate average depends on problem:

1) Hydraulic conductivity

$K(x)$ is often discontinuous



flow is $\perp$ to change in K

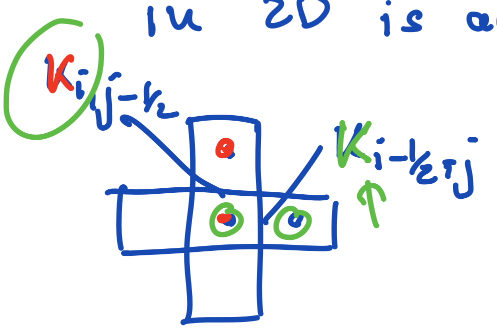
$\Rightarrow$ harmonic average

$$\boxed{K_{i-1/2} = \frac{2}{\frac{1}{K_{i-1}} + \frac{1}{K_i}}}$$

$\Delta b_i / \Delta l$

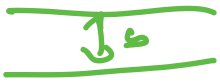
In 1D this gives <u>correct</u> solution

in 2D is an approx. but standard choice



2) Non-linear conductivity: $K = K(h)$

Examples: - unconfined flow



$$-\nabla \cdot [\overset{h}{b} K \nabla h] = f_s$$

- compressible flow

since h is smooth $\rightarrow$ $K(h)$ is smooth

$\rightarrow$ arithmetic average is best

Two options:

I) Evaluate then average

$$K_{i-1/2} = \frac{K(h_{i-1}) + K(h_i)}{2}$$

II Average & then evaluate

$$K_{i-1/2} = K\left(\frac{h_{i-1} + h_i}{2}\right)$$

~~seas~~ assumes h is smooth

Note: These averages are special case of general power law average

$$\boxed{K_p = \left(\frac{1}{2}\left(K_{i-1}^p + K_i^p\right)\right)^{1/p}}$$

$p=1$ arith.

$p=-1$ harm.

Implementation in comp_mean.m

From build_ops we have $\underline{M}$ matrix that computes arithmetic average!

As such we can compute averages as follows.

arithm. (p=1): $Kmean = \underline{M} * \underline{K}$

harmonic (p=0): $Kmean = 1./(\underline{M}.(1./\underline{K}))$

or simply use general power mean

$$\boxed{\underline{Kmean} = (\underline{M} * \underline{K}.\hat{\ }p).\hat{\ }(1/p)} \qquad p \neq 0$$

$\underline{\underline{Kd}}$ = spdiags(Kmean, 0, Nf, Nf)

spdiags (Kmean)   try ?