Introduction to melt migration

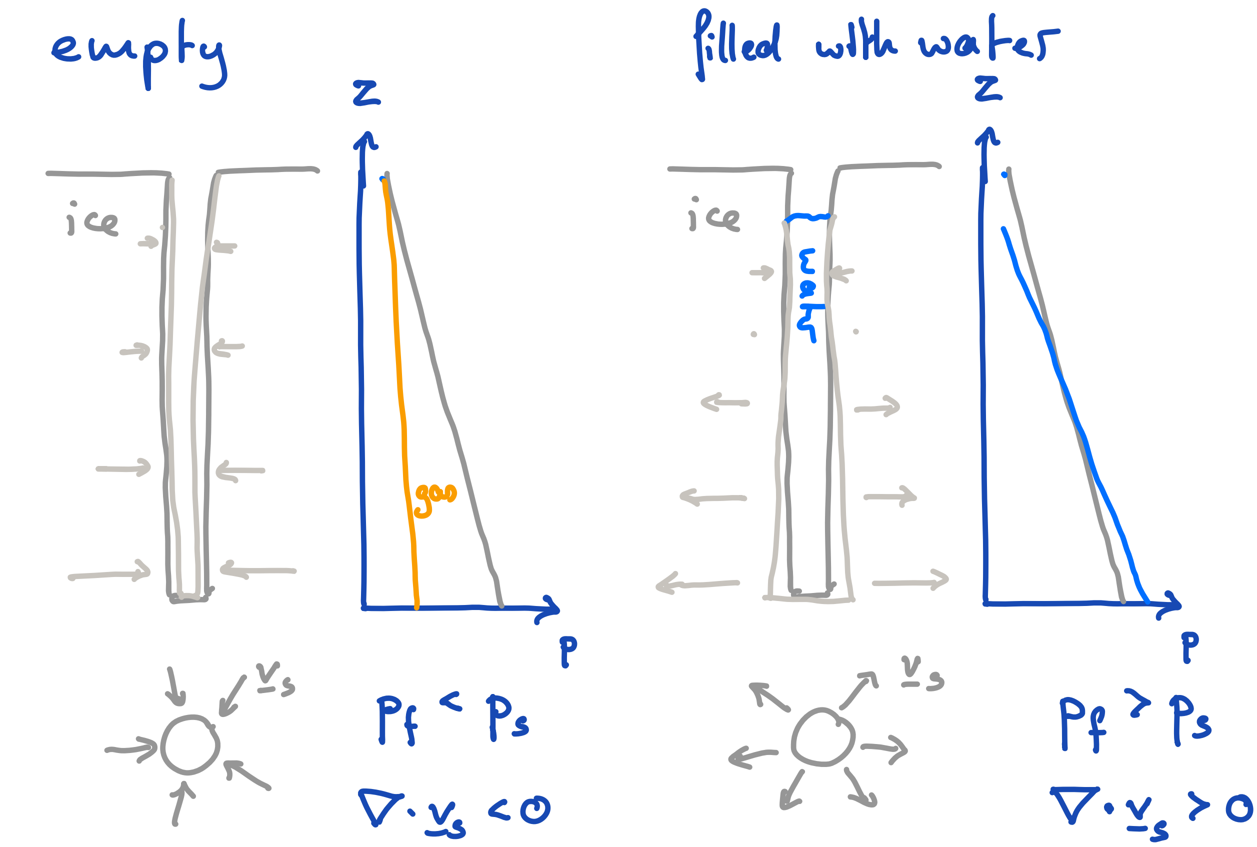
So far we have considered melt migration in a rigid rock, $\underline{v}_s = 0$. For class project we are interested in partially molten ice. Ice is not rigid and deforms by ductile creep. The simplest model for creep is to assume ice is a very viscous fluid.

How viscous?

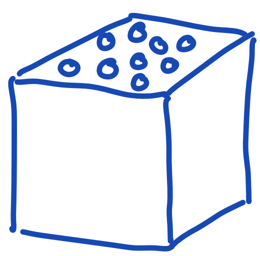
water	~	1 Pa s
ice	~	$10^{12} - 10^{14}$ Pa s

Key feature of viscous rheology is that it cannot support any stress. Deformation will continue as long as stress persists.

Consider a bore hole in ice (Nye 1953)



We can translate this to porous medium by imagining a block of ice with a set of small tubes, where we have the compaction relation



$$\boxed{p_f - p_s = \xi \, \nabla \cdot \underline{v}_s}$$

ξ = bulk or compaction viscosity

Note: Empirical relation similar to Darcy's law

Compaction viscosity : $\boxed{\xi = c\,\frac{\eta}{\phi^m}}$

η = shear viscosity

ϕ = porosity

m = exp. 0 – 1

c = coeff. ~1 – 10

Overpressure in fluid : $\boxed{p = p_f - p_s}$

Assumption: $p_s = p_0 + \rho_s g (z_0 - z)$

solid pressure is lithostatic

$\Rightarrow \quad \nabla p_s = -\rho_s g$

Reformulate Darcy's law in terms of overpressure

$$q_r = -\frac{k}{\mu_f}(\nabla p_f + \rho_f g \hat{z}) = -\frac{k}{\mu_f}(\underbrace{\nabla p_f - \nabla p_s}_{\nabla p} + \underbrace{\nabla p_s}_{-\rho_s g} + \rho_f g \hat{z})$$

$$\boxed{q_r = -\frac{k}{\mu_f}(\nabla p + \Delta\rho g \hat{z})} \qquad \Delta\rho = \rho_f - \rho_s > 0$$

Mass balance equations

fluid: $\frac{\partial}{\partial t}(\rho_f \phi) + \nabla \cdot [\rho_f \underline{v}_f \phi] = \Gamma$

solid: $\frac{\partial}{\partial t}(\rho_s (1-\phi)) + \nabla \cdot [\rho_s \underline{v}_s (1-\phi)] = -\Gamma$

where Γ is melting rate $\frac{M}{L^3 T}$

Assume: $\rho_f = \text{const.}$ $\rho_s = \text{const.}$

Divide by density and sum equations

$$\frac{\partial \phi}{\partial t} + \nabla \cdot [\underline{v}_f \phi] = \frac{\Gamma}{\rho_f}$$

$$-\frac{\partial \phi}{\partial t} + \nabla \cdot [\underline{v}_s (1-\phi)] = -\frac{\Gamma}{\rho_s}$$

$$\nabla \cdot [\underline{v}_f \phi + \underline{v}_s (1-\phi)] = \frac{\Gamma}{\rho_f} - \frac{\Gamma}{\rho_s} = \frac{\rho_s - \rho_f}{\rho_f \rho_s} \Gamma = -\frac{\Delta \rho}{\rho_f \rho_s} \Gamma$$

$$\underline{v}_f \phi + \underline{v}_s (1-\phi) = (\underline{v}_f - \underline{v}_s)\phi + \underline{v}_s = \underline{q}_r + \underline{v}_s$$

Two-phase continuity equation:

$$\boxed{\nabla \cdot [\underline{q}_r + \underline{v}_s] = -\frac{\Delta \rho}{\rho_f \rho_s} \Gamma}$$

Substitute two constitutive laws:

1) Darcy: $\underline{q}_r = -\frac{k}{\mu_f} (\nabla p + \Delta \rho g \hat{z})$

2) Compaction: $p = \xi \nabla \cdot \underline{v}_s$

So that we have

$$\boxed{-\nabla\cdot\left[\frac{k}{\mu_f}\left(\nabla p+\Delta\rho g\,\hat{z}\right)\right]+\frac{p}{\xi}=-\frac{\Delta\rho}{\rho_f\,\rho_s}\,\Gamma}$$

Simplify by introducing the overpressure head

$$\boxed{h=z+\frac{p}{\Delta\rho g}}\qquad\Rightarrow\qquad p=\Delta\rho g\,(h-z)$$

$$\nabla p=\Delta\rho g\,(\nabla h-\hat{z})$$

substituting into Darcy's law

$$\boxed{q_r=-K\,\nabla h}\qquad\text{where}\qquad\boxed{K=\frac{k\,\Delta\rho g}{\mu_f}}$$

Compaction term:

$$\nabla\cdot v_s=\frac{p}{\xi}=\frac{\Delta\rho g}{\xi}(h-z)=\frac{h-z}{\Xi}\qquad\boxed{\Xi=\frac{\xi}{\Delta\rho g}}$$

Capital Xi

Continuity in terms of overpressure head:

$$\boxed{-\nabla\cdot\left[K\,\nabla h\right]+\frac{h}{\Xi}=\frac{z}{\Xi}-\frac{\Delta\rho}{\rho_f\,\rho_s}\,\Gamma}$$

This is governing equation for the melt head.

$\Rightarrow$ modified Helmholtz equation

Porosity evolution

Because viscous rheology cannot support stress and in two-phase system there is always stress due to density difference ductile media experience large porosity changes.

$\Rightarrow$ evolve the porosity

Solid mass balance

$$-\frac{\partial \phi}{\partial t} + \nabla \cdot [(1-\phi)\, v_s] = -\frac{\Gamma}{\rho_s}$$

$$-\frac{\partial \phi}{\partial t} - \nabla \cdot [\underline{v}_s \phi] + \nabla \cdot \underline{v}_s = -\frac{\Gamma}{\rho_s}$$

subsitituting compaction relation $\nabla \cdot \underline{v}_s = \frac{P}{\xi}$

$$\boxed{\frac{\partial \phi}{\partial t} + \nabla \cdot [\underline{v}_s \phi] = \frac{P}{\xi} + \frac{\Gamma}{\rho_s}}$$

Advection equation with two source terms

- porosity moves with solid velocity
- over pressure ($p > 0$) generates porosity
- melting ($\Gamma > 0$) generates porosity

Porosity evolution in terms of over pressure head

$$\frac{p}{\xi} = \frac{h-z}{\Xi}$$

$$\boxed{\frac{\partial \phi}{\partial t} + \nabla \cdot [\underline{v}_s \phi] = \frac{h-z}{\Xi} + \frac{\Gamma}{\rho_s}}$$

Either way we need to determine $\underline{v}_s$!

Solid velocity field

Strickly we have to solve a compressible Stokes equation to determine $\underline{v}_s$ and p_s.
We have already assumed that p_s is lithostatic.
Now we introduce approximation that is valid in small porosity limit, $\phi \ll 1$.

Helmholtz decomposition:

$$\underline{v}_s = \underbrace{-\nabla u}_{\text{dilation}} + \underbrace{\nabla \times \underline{\Psi}}_{\text{shear}}$$

u = scalar potential

$\underline{\Psi}$ = vector potential

Assume that shear is negligible

$$\boxed{\underline{v}_s = -\nabla \mathcal{U}}$$

Substitute into compaction relation

$$\nabla \cdot \underline{v}_s = \frac{h-z}{\Xi} \quad \Rightarrow \quad \boxed{-\nabla^2 \mathcal{U} = \frac{h-z}{\Xi}}$$

The model for melt migration in ductile ice comprises 3 non-linear coupled PDE's:

$$\boxed{\begin{array}{ll} \left.\begin{array}{ll} 1) & -\nabla\cdot[K\nabla h] + \dfrac{h}{\Xi} = \dfrac{z}{\Xi} - \dfrac{\Delta\rho}{\rho_f \rho_s}\Gamma \\ 2) & -\nabla^2 \mathcal{U} = \dfrac{h-z}{\Xi} \end{array}\right\} \text{Flow problem} \\ \left.\begin{array}{ll} 3) & \dfrac{\partial\phi}{\partial t} + \nabla\cdot[\underline{v}_s\phi] = \dfrac{h-z}{\Xi} + \dfrac{\Gamma}{\rho_s} \end{array}\right\} \text{Transport problem} \end{array}}$$

with the constitutive functions:

$\underline{v}_s = -\nabla\mathcal{U}$ and $q_r = -K\nabla h$

$k = k_0\phi^n$ or $K = K_0\phi^n$ $\quad K_0 = \dfrac{k_0\Delta\rho g}{\mu_f}$ $\quad n\in[2,3]$

$\xi = \dfrac{c\eta}{\phi^m} = \dfrac{\xi_0}{\phi^m}$ or $\Xi = \dfrac{\Xi_0}{\phi^m}$ $\quad \Xi_0 = \dfrac{c\eta}{\Delta\rho g} = \dfrac{\xi_0}{\Delta\rho g}$ $\quad m\in[0,1]$