Non-dimensionalization of melt migration equations

Why do this?
1) Cleans up the equations
2) Identify governing parameters
3) Identify terms that can be dropped
4) Better scaling of equations
$\Rightarrow$ helps the numerics.

Governing equation

1) $$-\nabla\cdot[K_0\phi^n\nabla h] + \frac{\phi^m}{\Xi_0} h = \frac{\phi^m}{\Xi_0} z - \frac{\Delta\rho}{\rho_f\rho_s}\Gamma$$

2) $$-\nabla^2 u = \frac{\phi^m}{\Xi_0}(h-z)$$

3) $$\frac{\partial\phi}{\partial t} + \nabla\cdot[\underline{v}_s\phi] = \frac{\phi^m}{\Xi_0}(h-z) + \frac{\Gamma}{\rho_s}$$

on $x\in[0,L]$, $z\in[0,H]$ and $t\in[0,T]$

Scale all variables

independent variables: $\underline{x}_D = \frac{\underline{x}}{x_c}$ $\quad t_D = \frac{t}{t_D}$

primary dependent variables: $\phi_D = \frac{\phi}{\phi_c}$ $\quad h_D = \frac{h}{h_c}$ $\quad u_D = \frac{u}{u_c}$

secondary dependent variables: $\underline{v}_D = \frac{\underline{v}}{v_c}$ $\quad q_D = \frac{q_r}{q_c}$ $\quad \Gamma_D = \frac{\Gamma}{\Gamma_c}$

All variables with subscript 'D' are dimensionless and char. scales are chosen so that the magnitude of dim.less variables is order one.

What are these char. scales?

Some obvious external scales: $x_c \to H, L$

$t_c \to T$

Typically we choose internal scales suggested by the equations themselves – see below

Non-dimensionalize by substituting scaled var.

$\phi = \phi_c \phi_D, \quad t = t_c t_D, \quad \underline{x} = x_c \underline{x} \quad \ldots$

for example: $\frac{\partial \phi}{\partial t} = \frac{\partial(\phi_c \phi_D)}{\partial(t_c t_D)} = \frac{\phi_c}{t_c} \frac{\partial \phi_D}{\partial t_D}$

because t_c & ϕ_c are constants!

$$\nabla \cdot = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial(x_c x)}, \frac{\partial}{\partial(x_c y)}, \frac{\partial}{\partial(x_c z)}\right)$$

$$= \frac{1}{x_c}\left(\frac{\partial}{\partial x_D}, \frac{\partial}{\partial y_D}, \frac{\partial}{\partial z_D}\right) = \frac{1}{x_c} \nabla_D \cdot$$

Over pressure equation

$$-\nabla\cdot[K_0\phi^n\nabla h] + \frac{\phi^m}{\Xi_0}h = \frac{\phi^m}{\Xi_0}z - \frac{\Delta p}{\rho_f\rho_s}\Pi$$

substitute

$$\phi_D^m$$

$$-\frac{K_0\phi_c^m h_c}{x_c^2}\nabla_D\cdot[\phi_D^m\nabla_D h_D] + \frac{\phi_c^m h_c}{\Xi_0}\phi_D^m h_D = \frac{\phi_c^m x_c}{\Xi_0}z_D - \frac{\Delta p\,\Pi_c}{\rho_f\rho_s}\Pi_D$$

introduce char. conductivity and comp. viscosity

$$K_c = K_0\,\phi_c^m \qquad \Xi_c = \frac{\Xi_0}{\phi_c^m}$$

Set divergence term to unity by dividing by coefficient

$$-\nabla_D\cdot[\phi_D^m\nabla_D h_D] + \underbrace{\frac{x_c^2}{K_c\Xi_c}}_{\Pi_1}\phi_D^m h_D = \underbrace{\frac{x_c^3}{K_c\Xi_c h_c}}_{\Pi_2}\phi_D^m z_D - \underbrace{\frac{\Delta p\,\Pi_c x_c^2}{K_c h_c\rho_f\rho_s}}_{\Pi_3}\Pi_D$$

Three dimension less groupings Π_1, Π_2 & Π_3

$\Pi_1 = \frac{x_c^2}{K_c\Xi_c}$ assuming ϕ_c is known

this provides an internal length scale

$$\frac{x_c^2}{K_c\Xi_c} = 1 \Rightarrow x_c = \sqrt{K_c\Xi_c} = \sqrt{K_0\phi_c^n\frac{\Xi_0}{\phi_c^m}} = \sqrt{\frac{k_0\phi_c^n\xi_0}{\mu_f\,\phi_c^m}}$$

$$\Xi_0 = \frac{\xi_0}{\Delta\rho g} \qquad K_0 = \frac{k_0\Delta\rho g}{\mu_f}$$

introduce $\xi_c = \frac{\xi_0}{\phi_c^m}$ $\quad k_c = k_0\phi_c^n$

$$\Rightarrow \boxed{x_c = \sqrt{\frac{k_c\xi_c}{\mu_f}}}$$ compaction length

Compaction length is the internal length scale in melt migration. The physical interpretation is the distance over which changes in porosity can be communicated in partially molten material.

Once x_c is known we look to Π_2 for a head scale. Note that $x_c = \sqrt{K_c \Xi_c}$

$$\Pi_2 = \frac{x_c^3}{K_c \Xi_c h_c} = \frac{x_c}{h_c} \Rightarrow \boxed{h_c = x_c}$$

Finally, we use Π_3 to determine Γ_c

$$\Pi_3 = \frac{\Delta\rho\, \Gamma_c x_c^2}{K_c h_c \rho_f \rho_s} = \frac{\Delta\rho\, \Gamma_c x_c}{K_c \rho_f \rho_s} = 1 \Rightarrow \Gamma_c = \frac{K_c \rho_f \rho_s}{x_c \Delta\rho}$$

Note: If we had an interesting melting model that might suggest its own Γ_c but here go with this scale.

Dimensionless eqn for overpressure is

$$\boxed{-\nabla_D \cdot [\phi_D^n \nabla h_D] + \phi_D^m h_D = \phi_D^m z_D - \Gamma_D}$$

- removed all parameters from the equation
- check later if $h_D \sim 1$!

Equation for velocity potential

$$-\nabla^2 u = \frac{\Phi^m}{I_0}(h - z) \qquad u_D = \frac{u}{u_c}$$

$$-\frac{u_c}{X_c^2}\nabla_D^2 u_D = \underbrace{\frac{\phi_c^m}{I_0}}_{I_c} \phi_D^m (x_c h_D - x_c z_D) \qquad \text{where } h_c = x_c$$

$$-\nabla_D^2 u_D = \underbrace{\frac{x_c^3}{I_c u_c}}_{\Pi_4} \phi_D^m (h_D - z_D)$$

setting $\Pi_4 = \frac{x_c^3}{I_c u_c} = 1 \Rightarrow \quad u_c = \frac{x_c^3}{I_c} = \frac{K_c \cancel{I_c} x_c}{\cancel{I_c}} = K_c x_c$

$$\boxed{u_c = K_c x_c}$$

dimensionless equation

$$\boxed{-\nabla_D^2 u_D = \phi_D^m (h_D - z_D)}$$

This also implies a solid velocity scale

$$\underline{v}_s = -\nabla u \qquad \underline{v}_D = \frac{\underline{v}_s}{v_c}$$

$$v_c \underline{v}_D = -\frac{u_c}{x_c} \nabla_D u_D \quad \Rightarrow \quad \boxed{v_c = \frac{u_c}{x_c} = K_c}$$

Scale the porosity evolution equation

$$\frac{\partial \phi}{\partial t} + \nabla \cdot [\underline{v}_s \phi] = \frac{\phi^m}{\Xi_0}(h - z) + \frac{\Gamma}{\rho_s}$$

substitute $\phi = \phi_c \phi_D$ $\underline{v}_s = v_c \underline{v}_D$ ...

$$\frac{\phi_c}{t_c}\frac{\partial \phi_D}{\partial t_D} + \frac{v_c \phi_c}{x_c} \nabla_D \cdot [\underline{v}_D \phi_D] = \underbrace{\frac{\phi_c^m}{\Xi_0}}_{\Xi_c} x_c \, \phi_D^m (h_D - x_D) + \frac{\Gamma_c}{\rho_s} \Gamma_D$$

scale to accumulation term

$$\frac{\partial \phi_D}{\partial t_D} + \underbrace{\frac{v_c t_c}{x_c}}_{\Pi_5} \nabla_D \cdot [\underline{v}_D \phi_D] = \underbrace{\frac{x_c t_c}{\Xi_c \phi_c}}_{\Pi_6} \phi_D^m (h_D - x_D) + \underbrace{\frac{\Gamma_c t_c}{\rho_s \phi_c}}_{\Pi_7} \Gamma_D$$

These three parameters groups suggest time scales

1) Advective : $\Pi_5 = \frac{v_c t_c}{x_c} = 1 \Rightarrow t_c = t_A = \frac{x_c}{v_c} = \frac{x_c}{K_c}$

t_A time for solid to flow one compaction length

2) Compaction: $\Pi_6 = \frac{x_c t_c}{\Xi_c \phi_c} = 1 \Rightarrow t_c = t_C = \frac{\phi_c \Xi_c}{x_c}$

time for porosity change to propagate

one compaction length by compaction

3) Reactive: $\Pi_7 = \frac{\Gamma_c t_c}{\rho_s \phi_c} = 1 \Rightarrow t_c = t_R = \frac{\rho_s \phi_c}{\Gamma_c}$

time to change porosity by ϕ_c

via melting / freezing

If solid deformation is induced by melt migration

then solid advection is small. For now we don't

focus on reaction.

$\Rightarrow$ choose compaction time scale $t_C = \frac{\phi \Xi_c}{x_c}$

substitute into PDE

$$\frac{\partial \phi_D}{\partial t_D} + \underbrace{\frac{v_c \phi_c \Xi_c}{x_c^2}}_{\phi_c} \nabla_D \cdot (\phi_D v_D) = \phi_D^m (h_D - z_D) + \underbrace{\frac{\Gamma_c \Xi_c}{\rho_s x_c}}_{Da = \frac{\rho_f}{\Delta \rho}} \Gamma_D$$

so that dimensionless porosity evolution is

$$\boxed{\frac{\partial \phi_D}{\partial t_D} + \phi_c \nabla_D \cdot (\phi_D \underline{v}_D) = \phi_D^m (h_D - z_D) + Da\, \Gamma_D}$$

Dimensionless system of equations

1) $$-\nabla_D \cdot [\phi_D^n \nabla_D h_D] + \phi_D^m h_D = \phi_D^m z - \Gamma_D$$

2) $$-\nabla_D^2 u_D = \phi_D^m (h_D - z_D)$$

3) $$\frac{\partial \phi_D}{\partial t_D} + \phi_c \nabla_D \cdot [\underline{v}_D \phi_c] = \phi_D^m (h_D - z_D) + Da\, \Gamma_D$$

on the domain $x_D \in [0, \frac{L}{x_c}]$ $z_D \in [0, \frac{H}{x_c}]$

$t_D \in [0, \frac{T}{t_c}]$

Governing parameters: ϕ_c Da $\frac{H}{x_c}$ $\left(\frac{L}{x_c}\right)$