Melt migration - Fundamental analytic solutions

Dimensionless analytic solutions (dropping subcript D)

1) $-\nabla\cdot[\phi^n \nabla h] + \phi^m h = \phi^m z + \Gamma$

2) $-\nabla^2 u = \phi^m (h-z)$

3) $\frac{\partial\phi}{\partial t} + \phi_c \nabla\cdot[\underline{v}\phi] = \phi^m (h-z) + \mathcal{D}a\,\Gamma$

constitutive laws $\quad \underline{q} = -\phi^n \nabla h \qquad \underline{v} = -\nabla u$

$$p = h - z = \phi^m \nabla\cdot\underline{v}$$

Steady Exchange Flow without melting

"Exchange flow" means that ice & melt move in opposite directions: ice $\uparrow$ & melt $\downarrow$

no melting: Γ

steady state: $\phi = \text{const} = 1$

from equ 3: $\quad \phi_c \nabla\cdot\underline{v} = (h-z) = p$

substituting compaction relation

$$\phi_c\, p = p \Rightarrow p = 0 \Rightarrow \underline{v} = \text{const}$$

$\Rightarrow$ Equ3 is trivially satisfied

Eqn 1 reduces to

$$-\nabla\cdot[\cancelto{1}{\phi^n}\nabla h] + \cancelto{1}{\phi^n}(\underbrace{\cancelto{0}{h-z}}_{p}) = 0$$

$$-\nabla^2 h = 0$$

Eqn 2 reduces to $\quad -\nabla^2 u = 0$

Integrating twice: $\quad h = a_1 z + a_2$

linear

$$u = b_1 z + b_2$$

Typically we determine coefficients from BC's but here we imagine a infinite vertical domain.

Instead we know from Eqn3

$$p = h - z = 0 \Rightarrow a_1 z + a_2 - z = 0 \Rightarrow a_1 = 1 \quad a_2 = 0$$

$$\boxed{h = z}$$

From Darcy's law: $\quad q = -\cancelto{1}{\phi^n}\cancelto{1}{\frac{dh}{dz}} = -1 \qquad \boxed{q = -1}$

From continuity:

$$\nabla\cdot[\underline{q}_r + \underline{v}_s] = 0 \qquad \underline{q} = \frac{\underline{q}_r}{K_c} \qquad \underline{v} = \frac{\underline{v}_s}{K_c}$$

$$\nabla\cdot[\underline{q} + \underline{v}] = 0 \qquad \text{(dim.less)}$$

$$\frac{d}{dz}[q + v] = 0 \Rightarrow q + v = c = \text{const.}$$

This constant is also typically determined from BC's and sets the "net motion" in an exchange flow the net motion is zero, $c=0$.

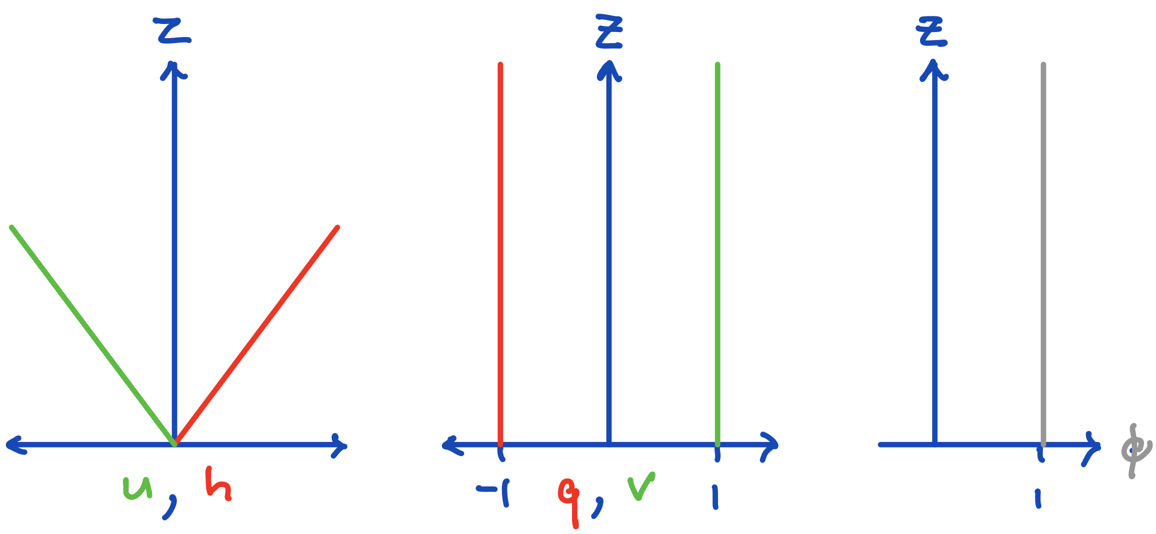
$\Rightarrow \boxed{v = -q = 1}$

Finally, the velocity potential is given b

$$v = -\frac{du}{dz} \quad \text{where } v=1 \quad \text{and} \quad u = b_1 z + b_2$$

$$1 = -b_1 \quad \Rightarrow \quad u = -z + c_1$$

here c_1 is arbitrary as only the gradient matters, $c_1 = 0$. $\boxed{u = -z}$



dimensionless solution:

$\phi = 1, \quad h = z, \quad u = -z, \quad q = -1, \quad v = 1$

This does not mean that ice and melt move at opposite & equal velocities!

q is a flux and v is a velocity.

Redimensionalize:

$$q_r = q_c q_0 = -K_c = -K_0 \phi_c^n \quad , \quad v_s = K_0 \phi_c^n$$

from def. of relative flux: $q_r = \phi(v_f - v_s)$

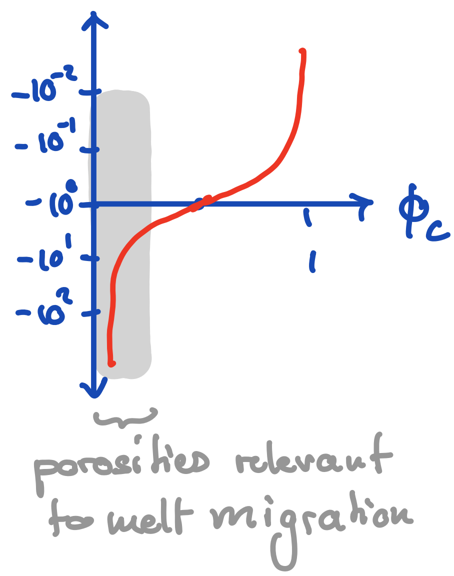
substituting: $-K_c = \phi_c (v_f - K_c)$

$$-\frac{K_c}{\phi_c} = v_f - K_c$$

$$v_f = K_c - \frac{K_c}{\phi} = K_c\left(1 - \frac{1}{\phi}\right)$$

$$v_f = K_c\left(\frac{\phi - 1}{\phi}\right) \qquad K_c = v_s$$

$$\boxed{\frac{v_f}{v_s} = -\frac{1-\phi}{\phi}}$$



At low porosities $|v_f| \gg |v_s|$!

In other words, the solid velocities induced by melt migration are typically small.

Instantaneous Compacting Column

Consider a vertical column of dimensionless height $H_D = \frac{H}{x_c}$ with constant porosity $\phi_c . (\phi_D = 1)$ and solid top and bottom boundaries $(v_s = q_r = 0)$

Compute the instantaneous flow problem (h, u, q, v).

The dimensionless form of eqns 1 & 2 is:

1) $-\frac{d^2h}{dz^2} + h = z$ with $q = -\frac{dh}{dz}\Big|_0 = -\frac{dh}{dz}\Big|_H = 0$

2) $-\frac{d^2u}{dz^2} = h - z$ with $v = -\frac{du}{dz}\Big|_0 = -\frac{du}{dz}\Big|_H = 0$

Equation 1 is non-homog. 2nd order ODE with constant coefficients. Solve by method of undetermined coefficients.

$h = h_h + h_p$ $\qquad h_h = c_1 e^{r_1 z} + c_2 e^{r_2 z}$

$h_p = c_3 z$

subsitute e^{rz} into homogeneous solution

$$-r^2 e^{rz} + e^{rz} = 0 \quad \Rightarrow \quad r^2 = 1 \qquad r = \pm 1$$

homogeneous solution : $h_h = c_1 e^{z} + c_2 e^{-z}$

substitute h_p into non-hom. equ

$$-\cancel{\frac{d^2 h_p}{dz^2}}^{\,0} + h_p = z \quad \Rightarrow \quad c_3 = 1 \qquad h_p = z$$

Full solution : $h = c_1 e^{z} + c_2 e^{-z} + z$

$$\frac{dh}{dz} = c_1 e^{z} - c_2 e^{-z} + 1$$

BC: $\left.\frac{dh}{dz}\right|_0 = c_1 - c_2 + 1 = 0 \quad \Rightarrow \quad c_2 = c_1 + 1$

$\left.\frac{dh}{dz}\right|_H = c_1 e^{H} - c_2 e^{-H} + 1 = 0$

solving for c_1 & c_2: $c_1 = \frac{e^{-H} - 1}{e^{H} - e^{-H}} \qquad c_2 = \frac{e^{H} - 1}{e^{H} - e^{-H}}$

Hence the solution is:

$$\boxed{\begin{aligned} h(z) &= z + \frac{e^{-H} - 1}{e^{H} - e^{-H}} e^{z} + \frac{e^{H} - 1}{e^{H} - e^{-H}} e^{-z} \\ q(z) = -\frac{dh}{dz} &= \underbrace{-1}_{\text{main solution}} + \underbrace{\frac{e^{-H} - 1}{e^{H} - e^{-H}} e^{z} - \frac{e^{H} - 1}{e^{H} - e^{-H}} e^{-z}}_{\text{boundary layers}} \end{aligned}}$$