Derivation of the equation of motion

Start with the general balance law.

$$\frac{\partial u}{\partial t} + \nabla \cdot j(u) = f_s$$

where u is the unkown that is balanced, $j(u)$ is a set of fluxes that transport the unknown and f_s is a set of source terms.

So far the unknown u has been a scalar, for example the energy $u = \rho c_p T$. Here we will consider an unknown that is a vector which will give the equations a tensorial nature.

The equations of motion are based on Euler's "Principle of linear momentum" which states that: The total force on a body is equal to the rate of change of the total momentum of the body. (1752)

Hence our unknown is the momentum, in particular the linear* momentum, $\underline{u} = \rho \underline{v}$, of the body, which is a vector. Here ρ is the density and $\underline{v}$ the velocity. Linear momentum is generated within the body by body forces, here we will only consider gravity $f_s = \rho \vec{g}$ where $\bar{g}$ is the gravitational acceleration.

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* The angular momentum will come into the derivation later.

Now we need to consider the fluxes of linear momentum (2)

into and out of a control volume.

a) Advective momentum flux

$$\underline{\underline{j}}_A = \underline{v} \otimes (\rho \underline{v}) = \rho \underline{v} \otimes \underline{v} = \rho \underline{v}\underline{v}^T$$

this is the linear momentum, $\rho \underline{v}$, advected by the velocity of the fluid, $\underline{v}$, itself. Hence, the advective momentum flux is inherently non-linear.

Note that $\underline{\underline{j}}_A$ is a 2nd order tensor, ie. a matrix. and $\otimes$ denotes the outer product as opposed to the inner product.

Inner product: $\underline{v} \cdot \underline{v} = \underline{v}^T \underline{v} = v_i v_i = v_1 v_1 + v_2 v_2 + v_3 v_3 = \text{scalar}$

Outer product: $\underline{v} \otimes \underline{v} = \underline{v}\,\underline{v}^T = v_i v_j = \begin{pmatrix} v_1 v_1 & v_1 v_2 & v_1 v_3 \\ v_2 v_1 & v_2 v_2 & v_2 v_3 \\ v_3 v_1 & v_3 v_2 & v_3 v_3 \end{pmatrix}$

b) Diffusive momentum flux:

Linear momentum can enter/exit a control volume even if the flow is parallel to the boundary by momentum diffusion due to shear and normal stresses

$\underline{\underline{j}}_D = -\underline{\underline{\sigma}}$ where $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}$ $\Rightarrow$ $\underline{\underline{j}}_D = p\underline{\underline{I}} - \underline{\underline{\tau}}$

where $\underline{\underline{\sigma}}$ is the Cauchy stress tensor, which can be decomposed into a volumetric stress, $-pI$, where p is the pressure of the fluid and the deviatoric stress, $\underline{\underline{\tau}}$.

Substituting these expressions into the general balance law we obtain the equations of motion

$$\boxed{\frac{\partial \rho \underline{v}}{\partial t} + \nabla \cdot \left[\rho\, \underline{v} \otimes \underline{v} + p\underline{\underline{I}} - \underline{\underline{\tau}}\right] = \rho g}$$ Cauchy Momentum Equation

This equation is very general and is the starting point for all of fluid mechanics. Next we need to complete the model by defining constitutive equations.

Incompressible Newtonian Fluid

Here we will consider an incompressible Newtonian fluid

For an incompressible fluid $\nabla \cdot \underline{v} = 0$ $\left(\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + \nabla \cdot (\underline{v}\rho) = 0 \right)$
(without source terms) $\rho = \text{const.}$

For a Newtonian fluid the deviatoris stress, $\underline{\underline{\tau}}$, depends linearly on the strain rate, $\dot{\underline{\underline{\varepsilon}}}$

$$\boxed{\underline{\underline{\tau}} = 2\mu\, \dot{\underline{\underline{\varepsilon}}}}$$ where μ = dynamic viscosity $\left[\frac{M}{LT}\right]$

rate of strain tensor $\dot{\underline{\underline{\varepsilon}}} = \frac{1}{2}\left(\nabla \underline{v} + \nabla^T \underline{v}\right)$

where $\nabla \underline{\underline{v}} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix}$ is the velocity gradient tensor

and $\nabla^T \underline{v} = (\nabla v)^T$ is its transpose.

Hence we have the deviatoric and full stress tensors

$$\underline{\underline{\tau}} = \mu\left(\nabla\underline{v} + \nabla^T\underline{v}\right)$$

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mu\left(\nabla\underline{v} + \nabla^T\underline{v}\right)$$

for an Newtonian fluid.

Substituting these expressions into Cauchy's Momentum Equation we obtain

$$\frac{\partial \underline{v}}{\partial t} + \nabla\cdot\left[\underline{v}\otimes\underline{v} + \frac{p}{\rho}\underline{\underline{I}} - \frac{\mu}{\rho}\left(\nabla\underline{v} + \nabla^T\underline{v}\right)\right] = \underline{g} \qquad \text{note: } \underline{g} = -g\hat{z}$$

Introduce $\nu = \frac{\mu}{\rho}$ $\left[\frac{L^2}{T}\right]$ kinematic viscosity or momentum diffusivity.

Also $\frac{1}{\rho}\nabla\cdot(p\underline{\underline{I}}) = \frac{1}{\rho}\nabla p$ we can write

$$\boxed{\frac{\partial \underline{v}}{\partial t} + \nabla\cdot[\underline{v}\otimes\underline{v}] = -\frac{\nabla p}{\rho} + \nabla\cdot\left[\nu\left(\nabla\underline{v} + \nabla^T\underline{v}\right)\right] + \underline{g}}$$

Navier-Stokes equation for variable viscosity ν.

In geopysical applications ν is strongly dependent on both temperature and strain rate.

For constant viscosity, we can take ν out of the divergence and simplify.

$$\nabla\cdot\left[\nabla\underline{v} + \nabla^T\underline{v}\right] = \nabla\cdot\nabla\underline{v} + \nabla(\overbrace{\cancel{\nabla\cdot\underline{v}}}^{0 \text{ incompressible}}) = \nabla^2\underline{v} \qquad \text{"vector Laplacian"}$$

Standard Navier-Stokes equation

$$\frac{\partial \underline{v}}{\partial t} + \nabla\cdot[\underline{v}\otimes\underline{v}] = -\frac{\nabla p}{\rho} + \nu\nabla^2\underline{v} + \underline{g} \quad \& \quad \nabla\cdot\underline{v} = 0 \qquad \text{conservative form}$$

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} = -\frac{\nabla p}{\rho} + \nu\nabla^2\underline{v} + \underline{g} \quad \& \quad \nabla\cdot\underline{v} = 0 \qquad \text{convective form}$$