Discretizing the Stokes Equation

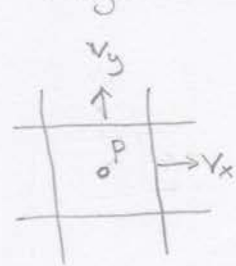
Dimensionless variable viscosity Stokes equation:

$$\boxed{\begin{aligned} 1)\quad & \nabla\cdot[\mu(\nabla\underline{v}+\nabla^T\underline{v})] - \nabla p = f_1 \\ 2)\quad & \nabla\cdot\underline{v} = f_2 \end{aligned}}$$

Overall we are looking for a discrete system of the form

$$\begin{bmatrix} \underline{\underline{A}} & \underline{\underline{C}}^T \\ \underline{\underline{C}} & 0 \end{bmatrix} \begin{bmatrix} \underline{v} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} \underline{f_1} \\ \underline{f_2} \end{bmatrix}$$

where $\underline{\underline{A}} = \underline{\underline{A}}^T$ ⇒ system is symmetric



Here $\underline{v} = \begin{bmatrix} \underline{v_x} \\ \underline{v_y} \end{bmatrix}$ on cell faces and $\underline{p}$ is a vector of cell center pressures.

First some low-hanging fruit:

From the discrete system

1) $\underline{\underline{A}} * \underline{v} + \underline{\underline{C}}^T * \underline{p} = \underline{f_1}$

2) $\underline{\underline{C}} * \underline{v} = \underline{f_2}$

From 2:

$\nabla\cdot\underline{v} \approx \underline{\underline{C}} * \underline{v} \Rightarrow \underline{\underline{C}} = \underline{\underline{D}}$ standard discrete divergence on our normal staggered grid

From 1:

$-\nabla p \approx \underline{\underline{C}}^T p \Rightarrow \underline{\underline{C}}^T = -\underline{\underline{G}}$ standard discrete gradient

$= -(-\underline{\underline{D}}^T) = \underline{\underline{C}}$ ✓

⇒ build on the existing staggered grid because it gets the relations between p and $\underline{v}$ right.

Divergence of deviatoric stress tensor

Cauchy stress : $\underline{\underline{\sigma}} = \underbrace{-p\underline{\underline{I}}}_{\text{vol.}} + \underbrace{2\mu\,\underline{\underline{\dot{\varepsilon}}}}_{\underline{\underline{\tau}}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}$

Deviatoric stress: $\underline{\underline{\tau}} = 2\mu\,\underline{\underline{\dot{\varepsilon}}} = \mu\left(\nabla\underline{v} + \nabla^T\underline{v}\right)$

$$\nabla\cdot\underline{\underline{\sigma}} = \nabla\cdot(\underline{\underline{\tau}} - p\underline{\underline{I}}) = \nabla\cdot\underline{\underline{\tau}} - \underbrace{\nabla p}_{\text{done}}$$

$\Rightarrow$ need to discretize divergence of deviatoric stress tensor.

Definition of the divergence of a 2nd order tensor:

$$\nabla\cdot\underline{\underline{\sigma}} = \sigma_{ij,j}\,\hat{e}_i = \begin{pmatrix} \sigma_{11,1} & \sigma_{12,2} \\ \sigma_{21,1} & \sigma_{22,2} \end{pmatrix} = \begin{pmatrix} \nabla\cdot(\sigma_{11}\ \sigma_{12}) \\ \nabla\cdot(\sigma_{21}\ \sigma_{22}) \end{pmatrix}$$

$\Rightarrow$ Divergence is applied row-wise to the tensor
1st-row is the divergence of the x-stresses
2nd-row is the divergence of the y-stresses

Definition of the gradient of a vector:

$$\nabla\underline{v} = v_{i,j}\,\hat{e}_i\otimes\hat{e}_j = \begin{pmatrix} v_{1,1} & v_{2,1} \\ v_{2,1} & v_{2,2} \end{pmatrix} = \begin{pmatrix} \nabla v_1^T \\ \nabla v_2^T \end{pmatrix} \qquad \text{hence} \quad \nabla^T\underline{v} = (\nabla\underline{v})^T = (\nabla v_1\,,\ \nabla v_2)$$

$\Rightarrow$ Gradient is also applied row-wise

Hence the symmetric velocity gradient / strain-rate tensor

$$\underline{\underline{\dot{\varepsilon}}} = \tfrac{1}{2}\left(\nabla\underline{v} + \nabla^T\underline{v}\right) = \begin{pmatrix} v_{1,1} & \tfrac{1}{2}(v_{1,2} + v_{2,1}) \\ \tfrac{1}{2}(v_{2,1} + v_{1,2}) & v_{2,2} \end{pmatrix} \quad\Rightarrow\quad \boxed{\underline{\underline{\dot{\varepsilon}}} = \underline{\underline{\dot{\varepsilon}}}^T} \ \text{symmetric}$$

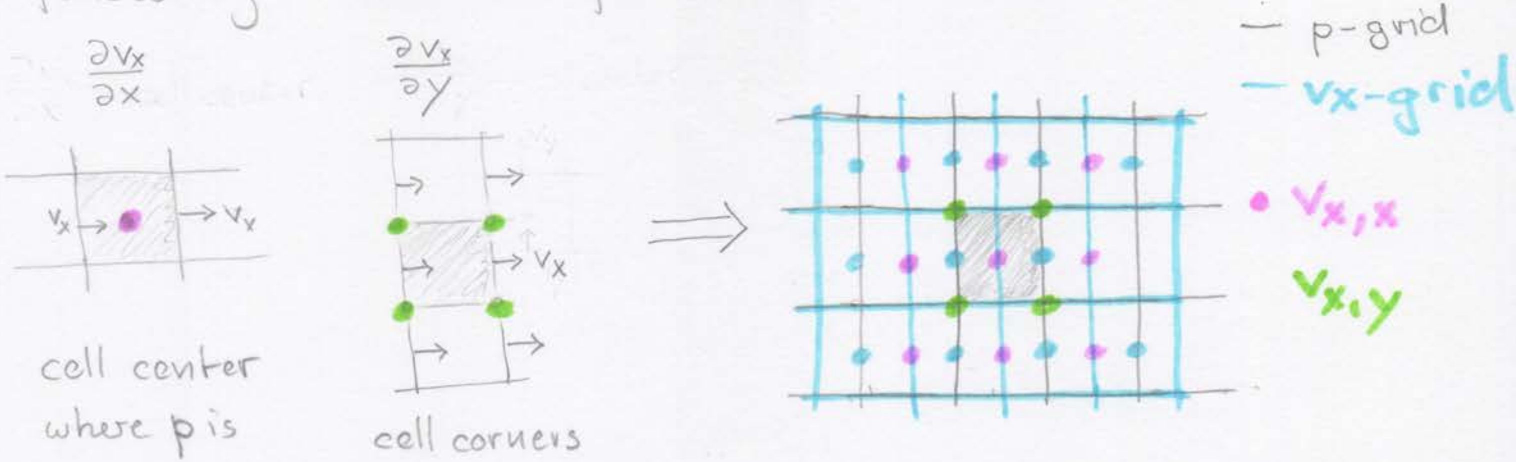
Discretizing the strain-rate tensor

To discretize the strain-rate tensor we need the following derivatives

$$v_{1,1} = \frac{\partial v_1}{\partial x_1} = \frac{\partial v_x}{\partial x} \qquad v_{1,2} = \frac{\partial v_1}{\partial x_2} = \frac{\partial v_x}{\partial y}$$

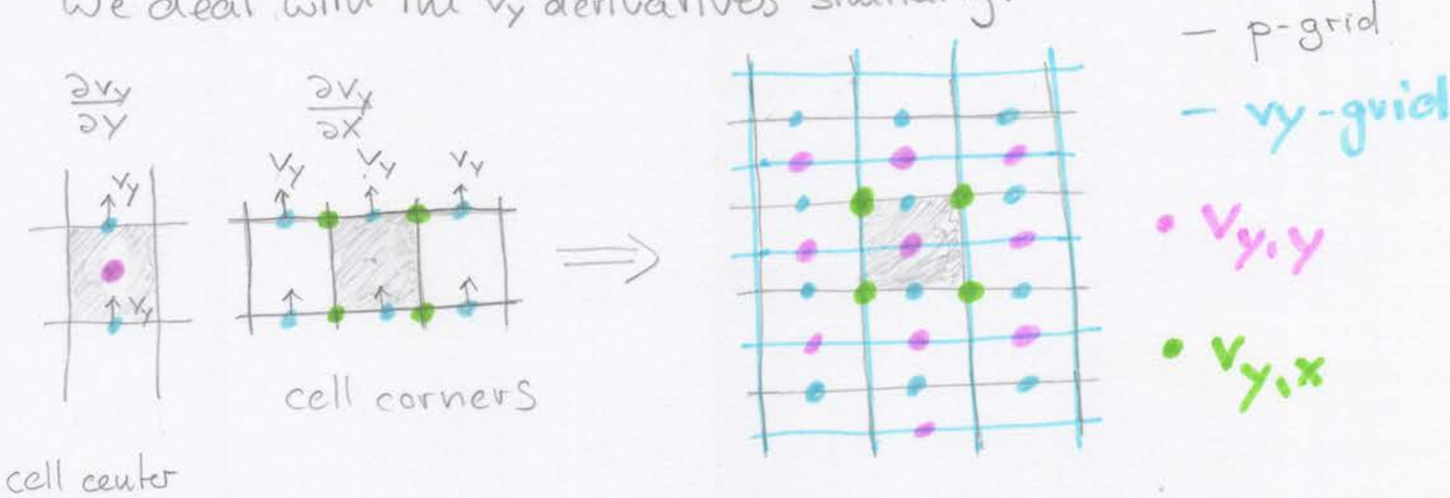
$$v_{2,2} = \frac{\partial v_2}{\partial x_2} = \frac{\partial v_y}{\partial y} \qquad v_{2,1} = \frac{\partial v_2}{\partial x_1} = \frac{\partial v_y}{\partial x}$$

Given the standard staggered grid we have the following locations for these derivatives.



⇒ Simplest way to compute the v_x-derivatives is to introduce a new grid centered on v_x. This vx-grid is shifted by $\frac{\Delta x}{2}$ relative to the p-grid and its size is $Nx+1$ by Ny if Nx and Ny are the size of the pressure grid.

We deal with the v_y derivatives similarly:



2D Stokes grid

- In 2D we use 3 staggered grids

1) Pressure grid: Primary grid that defines location of pressure and velocities
   Nx by Ny
   ⇒ used for heat transport calculations

2) X velocity grid: Shifted by $\frac{\Delta x}{2}$ in x-dir relative to pressure grid. Used to compute v_x derivatives in strain-rate tensor
   Nx+1 by Ny

3) Y velocity grid: Shifted by $\frac{\Delta y}{2}$ in y-dir relative to pressure grid. Used to compute v_y derivatives in strain-rate tensor.
   Nx by Ny+1

Note: In 3D we would have one additional grid for v_z velocity.

⇒ lots of bookkeeping.

New Matlab function:

Grid = build_stokes_grid(Gridp)

That builds all other grids given the primary pressure grid and returns a structure that holds all grids.

$$\left.\begin{array}{l}\text{Grid.p = pressure grid}\\ \text{Grid.x = x-velocity grid}\\ \text{Grid.y = y-velocity grid}\end{array}\right\}\ \begin{array}{l}\text{All three grids are built}\\ \text{using standard build\_grid.m}\\ \text{we have already developed}\end{array}$$

⇒ this will be on HW 9