Discretizing the divergence of the deviatoric stress

Term we need to discretize is

$$\nabla\cdot\left[\mu\left(\nabla\underline{v}+\nabla^T\underline{v}\right)\right] = \nabla\cdot\left(2\mu\,\underline{\underline{\dot{\varepsilon}}}\right) \approx \underbrace{\underline{\underline{D}} * 2\mu * \underline{\underline{Edot}}}_{\underline{\underline{A}}} * \underline{v} = \underline{\underline{A}} * \underline{v}$$

$\underline{\underline{D}}$ is the divergence of a tensor field

$\underline{\underline{Edot}}$ is a matrix that computes $\underline{\underline{\dot{\varepsilon}}}$ from $\underline{v}$

Discrete representation of strain-rate tensor

$$\underline{\underline{\dot{\varepsilon}}} = \begin{pmatrix} v_{1,1} & \frac{1}{2}(v_{1,2}+v_{2,1}) \\ \frac{1}{2}(v_{2,1}+v_{1,2}) & v_{2,2} \end{pmatrix} = \begin{pmatrix} \dot{\varepsilon}_{xy} & \dot{\varepsilon}_c \\ \dot{\varepsilon}_c & \varepsilon_{yy} \end{pmatrix} \Rightarrow \text{3 independent quantities}$$

How do we store $\underline{\underline{\dot{\varepsilon}}}$ as a function across the domain?

As a vector: $\underline{\text{eps\_dot}} = \begin{bmatrix} \underline{\text{eps\_dot\_xx}} \\ \underline{\text{eps\_dot\_yy}} \\ \underline{\text{eps\_dot\_c}} \end{bmatrix} = \underline{\underline{Edot}} * \underline{v}$

Here <u>eps_dot_xx</u> is a vector of all $\dot{\varepsilon}_{xx}$'s in all cell centers
<u>eps_dot_yy</u> is a vector of all $\dot{\varepsilon}_{yy}$'s in all cell centers
<u>eps_dot_c</u> is a vector of all $\dot{\varepsilon}_c$'s in all cell corners

This is similar to how we store the velocity vector where we don't store v_x & v_y for each cell together but we store component wise, ie., first all v_x's then all v_y's.

Now we need to find the entries into the block matrix $\underline{\underline{Edot}}$ which will comprise the discrete gradients on various grids.

To build Edot we need the discrete gradients on the x and y velocity grids

vx - grid: $\underline{\underline{G_x}} = \begin{bmatrix} \underline{\underline{G_{xx}}} \\ \underline{\underline{G_{xy}}} \end{bmatrix}$ vy - grid: $\underline{\underline{G_y}} = \begin{bmatrix} \underline{\underline{G_{yx}}} \\ \underline{\underline{G_{yy}}} \end{bmatrix}$

So that we can compute the partial derivatives as follows

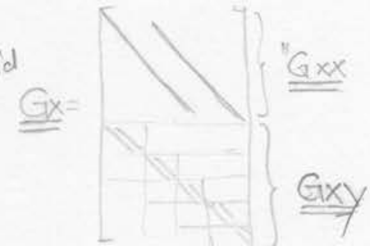
$$\frac{\partial v_x}{\partial x} = v_{x,x} = \underline{\underline{G_{xx}}} * \underline{v} \qquad \frac{\partial v_x}{\partial y} = v_{x,y} = \underline{\underline{G_{xy}}} * \underline{v}$$

$$\frac{\partial v_y}{\partial x} = v_{y,x} = \underline{\underline{G_{yx}}} * \underline{v} \qquad \frac{\partial v_y}{\partial y} = v_{y,y} = \underline{\underline{G_{yy}}} * \underline{v}$$

Note: build_grid will give you $\underline{\underline{G_x}}$ you have to extract the $\underline{\underline{G_{xx}}}$ and $\underline{\underline{G_{xy}}}$ submatrices.

note these are for v_x grid

$\underline{\underline{G_{xx}}} = \underline{\underline{G_x}}(1:Nfx, :)$ $\underline{\underline{G_{xy}}} = \underline{\underline{G_x}}(Nfx+1:Nf, :)$



We need to compute $\underline{eps\_dot} = \underline{\underline{Edot}} * \underline{v}$

$$\underline{\underline{\mathring{\varepsilon}}} = \begin{pmatrix} v_{x,x} & \frac{1}{2}(v_{x,y}+v_{y,x}) \\ \frac{1}{2}(v_{x,y}+v_{y,x}) & v_{y,y} \end{pmatrix} \sim \begin{bmatrix} \underline{eps\_dot\_xx} \\ \underline{eps\_dot\_yy} \\ \underline{eps\_dot\_c} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\underline{G_{xx}}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{G_{yy}}} \\ \frac{1}{2}\underline{\underline{G_{xy}}} & \frac{1}{2}\underline{\underline{G_{yx}}} \end{bmatrix}}_{\underline{\underline{Edot}}} \begin{bmatrix} \underline{vx} \\ \underline{vy} \end{bmatrix}$$

We need to know the size of the zero blocks!

$$\boxed{\underline{\underline{Edot}} = \begin{bmatrix} \underline{\underline{G_{xx}}} & \underline{\underline{Z_{xy}}} \\ \underline{\underline{Z_{yx}}} & \underline{\underline{G_{yy}}} \\ \frac{1}{2}\underline{\underline{G_{xy}}} & \frac{1}{2}\underline{\underline{G_{yx}}} \end{bmatrix}}$$

where the zero blocks are

$\underline{\underline{Z_{xy}}}$ is Grid.x.Nfx by Grid.y.N

$\underline{\underline{Z_{yx}}}$ is Grid.y.Nfy by Grid.x.N

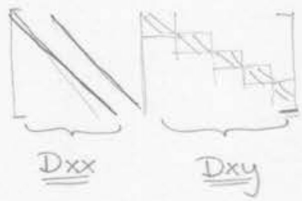
and of course they are sparse. ⇒ spalloc

The deviatoric stress $\underline{\underline{\tau}}$ is now simply

$$\underline{\mathrm{tau}} = \begin{bmatrix} \mathrm{tau\_xx} \\ \mathrm{tau\_yy} \\ \mathrm{tau\_c} \end{bmatrix} = 2\mu\, \underline{\underline{\mathrm{Edot}}} * \underline{v}$$

To complete the assembly of the $\underline{\underline{A}}$ matrix we need to take the divergence of $\underline{\mathrm{tau}}$. To do this we need the x and y submatrices of the divergence operators on the v_x and v_y grids.

$\underline{\underline{Dx}}$ =



$\underline{\underline{Dxx}} = \underline{\underline{Dx}}(:, 0 : Nfx)$ $\quad$ $\underline{\underline{Dxy}} = \underline{\underline{Dx}}(:, Nfx+1 : Nf)$

similarly for $\underline{\underline{Dyx}}$ and $\underline{\underline{Dyy}}$

We need to discretize:

$$\nabla \cdot \underline{\underline{\tau}} = \begin{bmatrix} \nabla\cdot(\tau_{xx}, \tau_{xy}) \\ \nabla\cdot(\tau_{yx}, \tau_{yy}) \end{bmatrix} = \begin{bmatrix} \tau_{xx,x} + \tau_{xy,y} \\ \tau_{yx,x} + \tau_{yy,y} \end{bmatrix} \approx \underbrace{\left[\begin{array}{c|c|c} \underline{\underline{Dxx}} & \underline{\underline{Zyx}}^T & \underline{\underline{Dxy}} \\ \hline \underline{\underline{Zxy}}^T & \underline{\underline{Dyy}} & \underline{\underline{Dyx}} \end{array}\right]}_{\underline{\underline{D}}} \begin{bmatrix} \mathrm{tau\_xx} \\ \mathrm{tau\_yy} \\ \mathrm{tau\_c} \end{bmatrix}$$

Hence the $\underline{\underline{A}}$ matrix is given by: $\underline{\underline{A}} = 2\mu\, \underline{\underline{D}} * \underline{\underline{\mathrm{Edot}}}$

$$\underline{\underline{A}} = 2\mu \begin{bmatrix} \underline{\underline{Dxx}} & \underline{\underline{Zyx}}^T & \underline{\underline{Dxy}} \\ \underline{\underline{Zxy}}^T & \underline{\underline{Dyy}} & \underline{\underline{Dyx}} \end{bmatrix} \begin{bmatrix} \underline{\underline{Gxx}} & \underline{\underline{Zxy}} \\ \underline{\underline{Zyx}} & \underline{\underline{Gyy}} \\ \frac{1}{2}\underline{\underline{Gxy}} & \frac{1}{2}\underline{\underline{Gyx}} \end{bmatrix} =$$

$$\underline{\underline{A}} = 2\mu \begin{bmatrix} \underline{\underline{Dxx}} * \underline{\underline{Gxx}} + \frac{1}{2}\underline{\underline{Dxy}} * \underline{\underline{Gxy}} & \frac{1}{2}\underline{\underline{Dxy}} * \underline{\underline{Gyx}} \\ \frac{1}{2}\underline{\underline{Dyx}}\, \underline{\underline{Gxy}} & \underline{\underline{Dyy}} * \underline{\underline{Gyy}} + \frac{1}{2}\underline{\underline{Dyx}} * \underline{\underline{Gyx}} \end{bmatrix}$$

$\underline{\underline{A}} = \underline{\underline{A}}^T$ in the interior on bnd the natural BC's in $\underline{\underline{G}}$ cause deviations

We will write a function to build these operators

$[\underline{\underline{D}}, \underline{\underline{Edot}}, \underline{\underline{Dp}}, \underline{\underline{Gp}}, \underline{\underline{Z}}, \underline{\underline{I}}]$ = build_stokes_ops(Grid

$$\underline{\underline{D}} = \begin{bmatrix} \underline{\underline{Dxx}} & \underline{\underline{Zyx}}^T & \underline{\underline{Dxy}} \\ \underline{\underline{Zxy}}^T & \underline{\underline{Dyy}} & \underline{\underline{Dyx}} \end{bmatrix}$$

divergence of a tensor (matrix)

$$\underline{\underline{Edot}} = \begin{bmatrix} \underline{\underline{Gxx}} & \underline{\underline{Zxy}} \\ \underline{\underline{Zyx}} & \underline{\underline{Gyy}} \\ \frac{1}{2}\underline{\underline{Gxy}} & \frac{1}{2}\underline{\underline{Gyx}} \end{bmatrix}$$

symmetric gradient of velocity

$\underline{\underline{Dp}}$, $\underline{\underline{Gp}}$ are the standard discrete div & grad ops on the primary pressure grid

$\underline{\underline{Z}}$ is an all sparse zero matrix for the lower right diagonal block in the stokes system

$\underline{\underline{I}}$ is an $(Nf + N)$ by $(Nf$ by $N)$ identity for the implementation of the boundary conditions