Balance Laws

- Most fundamental equations in natural sciences are balance laws or derived from them.
- Balance law accounts for gains and losses of a quantity due to transport and sources/sinks.
- If there are no sinks/sources of a quantity it is a "conserved quantity" and the balance law becomes a conservation law.
- In multiphase systems like porous media It is not trivially obvious what quantities are conserved (in absence of obvious sorces/sinks like wells)
  - Mass of the pore fluid is conserved
  - Energy of the pore-fluid is not conserved
    $\rightarrow$ because fluid can loose/gain energy to/from grains
  - Total energy of the system is conserved.

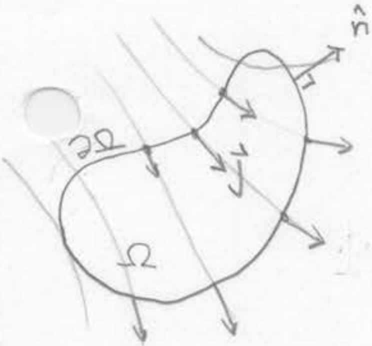
Note on units:

In the derivation of conservation laws it is important to ensure the units match. We use general notation:

$L \equiv$ units of length $\qquad N \equiv$ units of number eg mol
$T \equiv$ units of time $\qquad 1 \equiv$ unit less
$M \equiv$ units of mass $\qquad \# \equiv$ arbitrary units "stuff"

Indicate units of a quantity in brackets, e.g., viscosity $\mu \left[\frac{M}{LT}\right]$

Derivation of general balance law



Account for change of unknown $u(\underline{x},t)$ in domain Ω due to fluxes $\vec{j}$ across the boudary $\partial\Omega$ and production/consumption by f in Ω.

Units of basic quantities:

- u is a density $\left[\frac{\#}{L^3}\right]$
- $\vec{J}$ is a flux $\left[\frac{\#}{L^2 T}\right]$
- $\hat{f}_s$ is a volumetric rate $\left[\frac{\#}{L^3 T}\right]$

General balance on Ω: $\boxed{\frac{d}{dt} U = J + F}$

1) U is amount of u in Ω: $U(t) = \int_\Omega u(x,t)\, dV$ $\quad [\#]$

2) J is rate of transport of transport of u across $\partial\Omega$ by $\vec{J}$: $J(t) = -\oint_{\partial\Omega} j(u)\cdot\hat{n}\, dA$ $\quad \left[\frac{\#}{T}\right]$

3) F is rate of production/consumption in Ω: of u inside Ω: $F(t) = \int_\Omega \hat{f}_s(\underline{x},t)\, dV$ $\quad \left[\frac{\#}{T}\right]$

Substitute into balance equation:

$$\boxed{\frac{d}{dt}\int_\Omega u\, dV = -\oint_{\partial\Omega} \vec{j}(u)\cdot\hat{n}\, dA + \int_\Omega \hat{f}_s\, dV}$$ Integral balance law

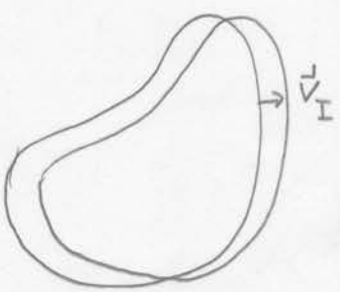
units: $\frac{1}{T}\frac{\#}{\cancel{L^3}}\cancel{L^3} = \frac{\#}{\cancel{L^2}T}\, 1\, \cancel{L^2} = \frac{\#}{\cancel{L^3}T}\cancel{L^3}$

$\Rightarrow$ each term $\left[\frac{\#}{T}\right]$ ie rate of change of # !

To obtain a local PDE we need to

1) Exchange derivative and integral
2) Transform surface to volume integral

1) Reynolds Transport Theorem:



The domain Ω is moving with velocity $\vec{v}$

$$\frac{d}{dt}\int_{\Omega(t)} u \, dV = \int_{\Omega} \frac{\partial u}{\partial t} dV + \oint_{\partial\Omega} u\,(\vec{v}_I \cdot \hat{n})\, dS$$

We consider Eulerian limit of fixed domain $\vec{v}_I = 0 \Rightarrow \frac{d}{dt}\int_{\Omega} u\,dV = \int_{\Omega} \frac{\partial u}{\partial t} dV$

2) Divergence theorem: $\oint_{\partial\Omega} \vec{j}\cdot\hat{n}\, dA = \int_{\Omega} \nabla\cdot\vec{j}\, dV$

substitute: $\int_{\Omega}\left(\frac{\partial u}{\partial t} + \nabla\cdot\vec{j}(u) - \hat{f}_s\right) dV = 0$

Since this must hold for any domain Ω ! $\Rightarrow$ Integrand $= 0$

Local form of general balance law

$$\boxed{\frac{\partial u}{\partial t} + \nabla\cdot j(u) = \hat{f}_s}$$

where
u = unknown $\left[\frac{\#}{L^3}\right]$
$j(u)$ = flux $\left[\frac{\#}{L^2 T}\right]$
$\hat{f}_s$ = vol. rate $\left[\frac{\#}{L^3 T}\right]$