# Groundwater recharge between two rivers

```
clear, clc
set_demo_defaults();
```

Consider the following problem taken from Sanford et al. (2001) and shown below. The Danube-Tisza Interfluve is a large agricultural area in Hungary between two the Danube and the Tisza rivers. The mean elevation of the Danube in the area is approximately 90 m and the mean elevation of the Tisza in the area is apporximately 80 m above sea level. The region recieves an average of 1.5 cm/yr of precipitation and the underlying aquifer is approximately 100 m thick with a hydraulic conductivity of approximately 2×10^{-2} cm/s.
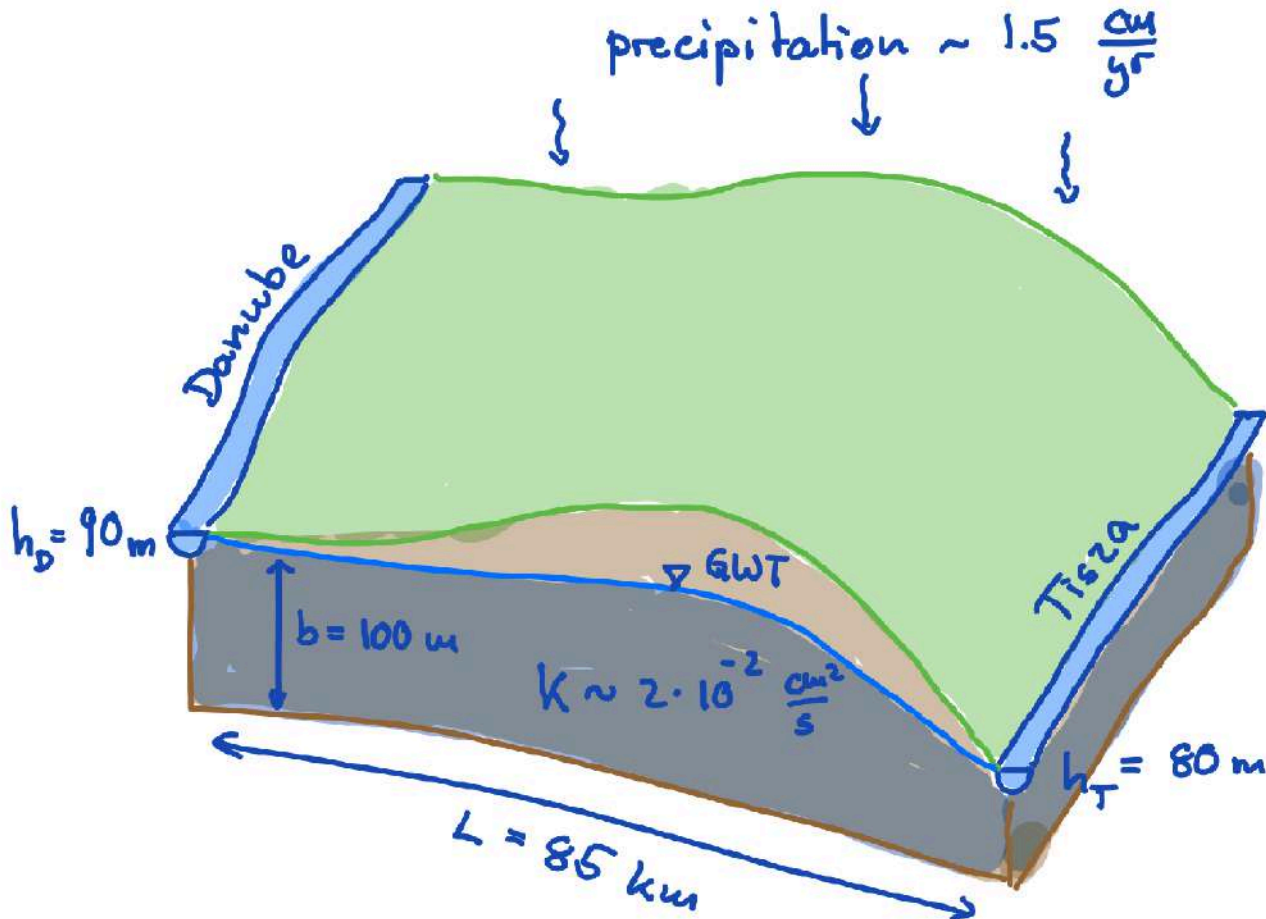


```
%% Physical properties
cm2m = 1/100;          % cm to m conversion
yr2s = 365*24*60^2;    % yr to s conversion

Length = 85070;        % Distance between Danube and Tisza rivers [m]
Width = 5430;          % Width of segment considered [m]
K =2e-2*cm2m;          % Hydraulic conductivity [m/s]
qp = 1.5*cm2m/yr2s;    % Average annual precipitation [m3/m2/s]
hD = 90;               % Elevation of Danube river[m]
```

```
hT = 80;              % Elevation of Tisza river [m]
b = 100;              % Aquifer thickness [m]
```

The area is popuar for hydrological studies because the rivers run parallel for more than 100 km (see map) with very little change in elevation. This setup a geometry that can be approximated by a two-dimensional cross-section.
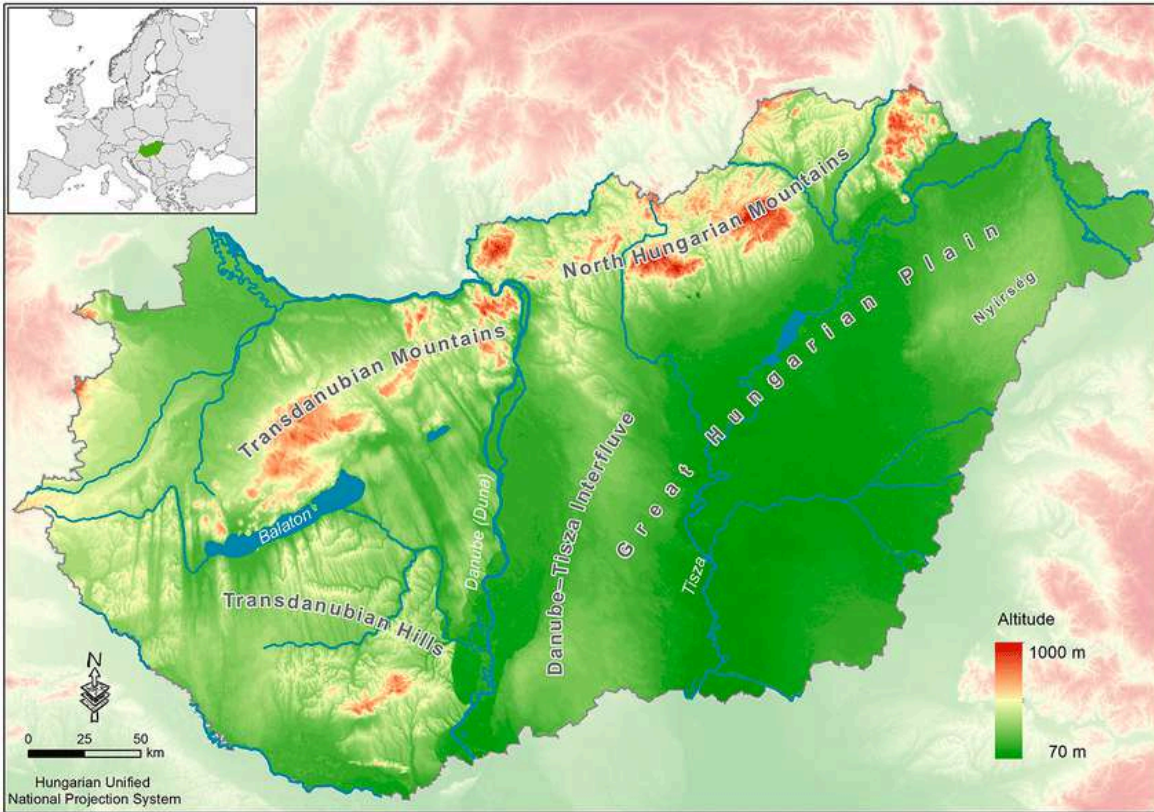
Shallow aquifer model

We further simplify the model by introducing a shallow aquifer approximation because the large aspect ratio of cross section, $b/H = 1/850 \approx 10^{-3}$. In this limit the flow is horizontal to first-order and we can integrate the the governing equation as follows

$$\int_0^{b(x)} \nabla \cdot \mathbf{q}\,dz = \nabla_h \cdot \int_0^{b(x)} \mathbf{q}_h\,dz + \mathbf{q} \cdot \hat{\mathbf{n}}_{\text{top}}\big|_{z=b(x)} - \mathbf{q} \cdot \hat{\mathbf{n}}_{\text{bot}}\big|_{z=0}$$

Making the following assumptions

1. The flux is horizontal: $\mathbf{q}_h = [q_x\ 0]^T$ and $q_x = -K\dfrac{dh}{dx}$, where pressure is hydrostatic so that $h = h(x)$

2. The base of the aquifer is impermable: $\mathbf{q} \cdot \hat{\mathbf{n}}_{\text{bot}}\big|_{z=b(x)} = 0$

3. Slope of the top of the aquifer is negligible: $\mathbf{q} \cdot \hat{\mathbf{n}}_{\text{top}}\big|_{z=0} = -q_p$

We obtain the shallow aquifer model for the interfluve:

PDE: $-\widetilde{\nabla} \cdot \left[bK\widetilde{\nabla}h \right] = q_r$ on $x \in [0, L]$

BC: $h(0) = h_D = 90\,\text{m}$ and $h(x = L) = h_T = 80$ m

Here the tilde on the divergence and gradient operators indicates that they only contain horizontal derivatives, because we have integrated in the vertical.

In 1D the PDE can also simply be written as

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[bK\frac{\mathrm{d}h}{\mathrm{d}x}\right] = q_p$$

and solved by integrating twice to obtain the following solutions for the head h and flux q

$$h = h_D + \left(\frac{q_p L}{2bK} - \frac{h_D - h_T}{L}\right)x - \frac{q_p}{2bK}x^2$$

$$q = \frac{q_p}{b}\left(x - \frac{L}{2}\right) + \frac{K}{L}(h_D - h_T)$$

```
% Analytic solution
xa = linspace(0,Length,1e2);
ha = @(x,qp) hD + (qp*Length/2/b/K-(hD-hT)/Length)*x - qp/2/b/K*x.^2;
qa = @(x,qp) qp/b*(x-Length/2) + K/Length*(hD-hT);
```

Critical recharge to create a groundwater divide

In the absence of recharge water flows down the hydraulic gradient from the Danube towards the Tisza river and the head varies linearly. As recharge/precipitation increases the head is curved downwards and eventually a maximum forms that divides the intefluve into a region that flow into the Danume and another that flows into the Tisza river.

The location of the groundwater divide is obtained by setting $q = 0$ and solving for x to obtain

$$x_{\text{max}} = \frac{2}{L} + \frac{Kb(h_D - h_T)}{q_p L}$$

and the critical recharge is given when $x_{\text{max}} = 0$ so that

$$q_{\text{p,crit}} = \frac{2Kb(h_D - h_T)}{L^2}$$

```
xmax = @(qp) Length/2 - (K*b*(hD-hT))/(qp*Length);
qp_crit = 2*K*b*(hD-hT)/Length^2
```

```
qp_crit = 5.5272e-11
```

The model is shown below for increasing recharge/precipitation.

```
col = marc_colors();
% create custom color map
```

```matlab
N = 6;
RED = repmat(col.red,N,1);
BLUE = repmat(col.blue,N,1);
ALPHA = repmat(linspace(0,1,N)',1,3);
COL = BLUE.*ALPHA + RED.*(1-ALPHA);

figure('position',[10 10 900 600])
subplot 211
plot(xa/1e3,ha(xa,0e-11),'-','color',COL(1,:)), hold on
plot(xa/1e3,ha(xa,2e-11),'-','color',COL(2,:))
plot(xa/1e3,ha(xa,4e-11),'-','color',COL(3,:))
plot(xa/1e3,ha(xa,6e-11),'-','color',COL(4,:))
plot(xa/1e3,ha(xa,8e-11),'-','color',COL(5,:))
plot(xa/1e3,ha(xa,1e-10),'-','color',COL(6,:))
plot(xmax(6e-11)/1e3,ha(xmax(6e-11),6e-11),'o','markeredgecolor',COL(4,:),'markerfaceco
plot(xmax(8e-11)/1e3,ha(xmax(8e-11),8e-11),'o','markeredgecolor',COL(5,:),'markerfaceco
plot(xmax(1e-10)/1e3,ha(xmax(1e-10),1e-10),'o','markeredgecolor',COL(6,:),'markerfaceco
xlabel 'distance [km]'
ylabel 'head [m]'

subplot 212
plot([0 Length]/1e3,[0 0],'k-'), hold on
plot(xa/1e3,qa(xa,0e-11)/(cm2m/yr2s),'-','color',COL(1,:))
plot(xa/1e3,qa(xa,2e-11)/(cm2m/yr2s),'-','color',COL(2,:))
plot(xa/1e3,qa(xa,4e-11)/(cm2m/yr2s),'-','color',COL(3,:))
plot(xa/1e3,qa(xa,6e-11)/(cm2m/yr2s),'-','color',COL(4,:))
plot(xa/1e3,qa(xa,8e-11)/(cm2m/yr2s),'-','color',COL(5,:))
plot(xa/1e3,qa(xa,1e-10)/(cm2m/yr2s),'-','color',COL(6,:))
plot(xmax(6e-11)/1e3,0,'o','markeredgecolor',COL(4,:),'markerfacecolor','w','markersize
plot(xmax(8e-11)/1e3,0,'o','markeredgecolor',COL(5,:),'markerfacecolor','w','markersize
plot(xmax(1e-10)/1e3,0,'o','markeredgecolor',COL(6,:),'markerfacecolor','w','markersize
xlabel 'distance [km]'
ylabel 'flux [cm/yr]'
```
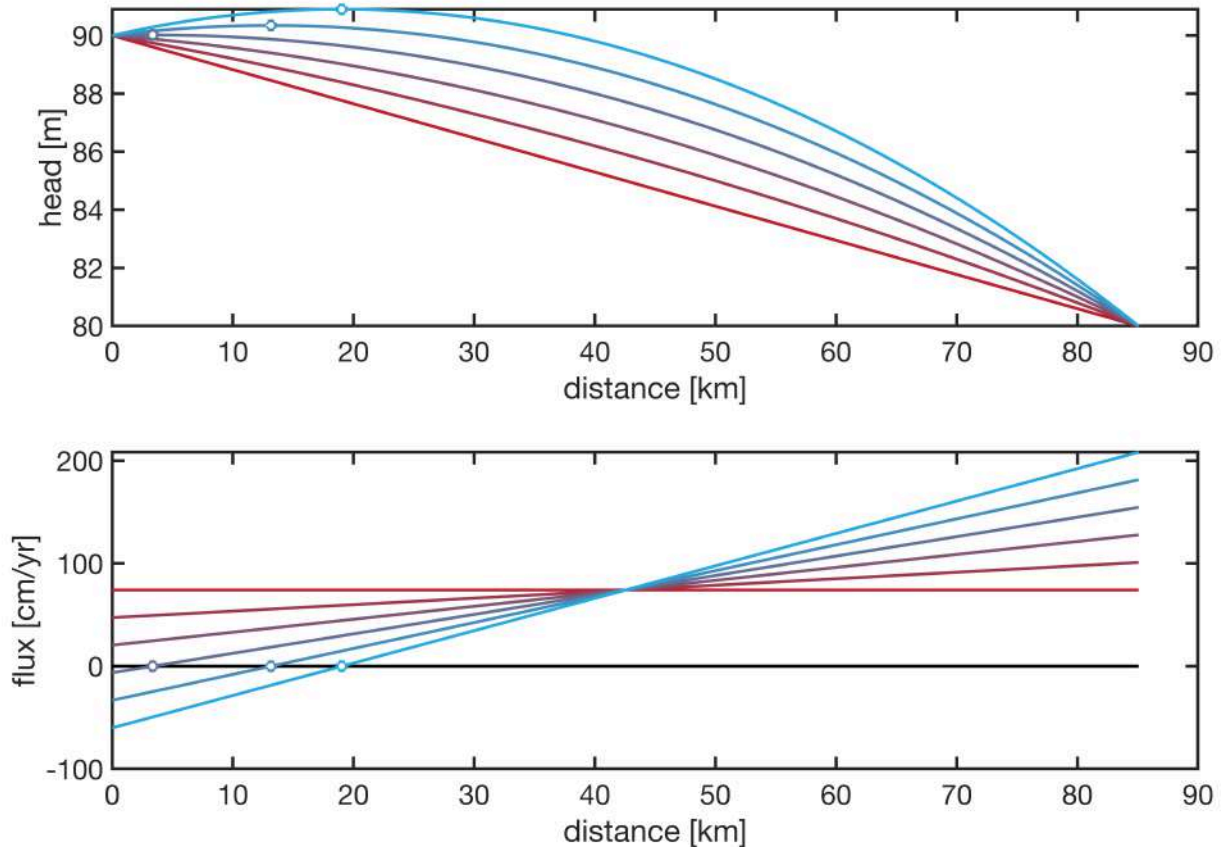
Comparison of simple model with real data

Our model is admittedly very simple, so a reality check is in order. Below are data of the real topographic cross section and groundwater table from the Danube-Tisza Interfluve

```
%% Topography and groundwater table: Danube - Tisza Rivers
mm2m = 1.81e3; % mm on map to m in reality
dist_topo = [ 0 14  17  19  21 23  25  28  32  33  35  37  42 43 44 47]*mm2m;
topo = [90 95 100 115 115 110 115 120 115 110 105 100 95 90 85 80];
dist_gw = [ 0 15 18 21 25 28 34 36 39 43 44 47]*mm2m;
gw = [90 95 100 105 110 110 105 100 95 90 85 80];
```
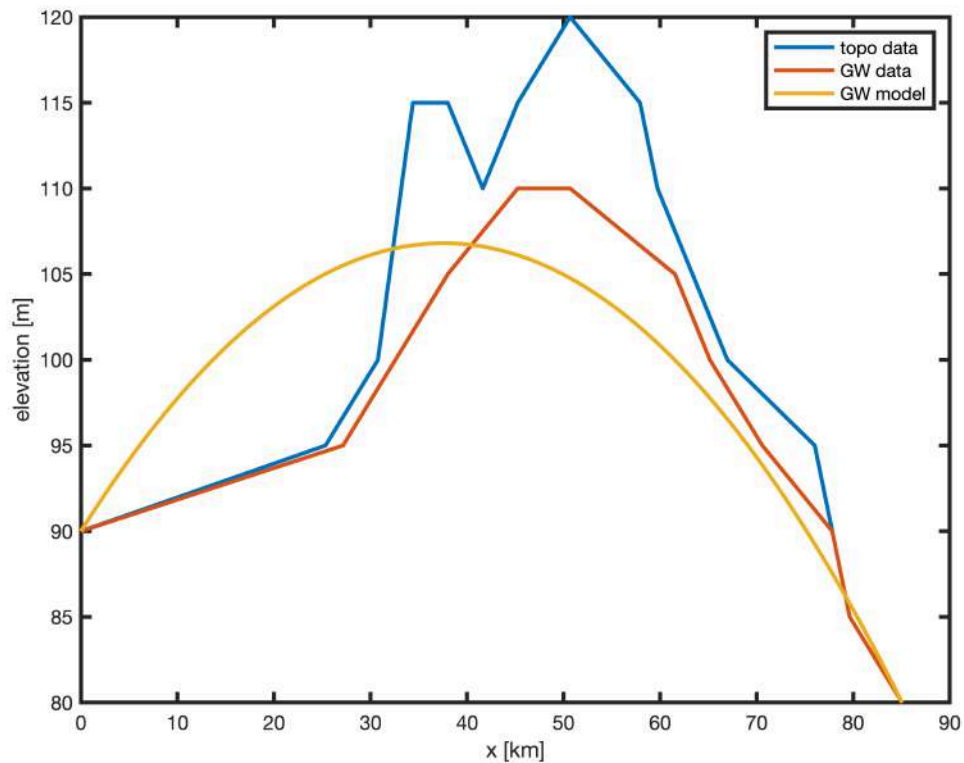
And here is the comparison with the model

```
figure
plot(dist_topo/1e3,topo,'-'), hold on
plot(dist_gw/1e3,gw,'-')
plot(xa/1e3,ha(xa,qp),'-')
legend('topo data','GW data','GW model')
xlabel('x [km]')
ylabel('elevation [m]')
```

Well, you gotta start somewhere! Clearly the groundwater table interacts with the topography on the left side, which changes the pattern.

Numerical solution

A preview of the numerical solution to our simplified problem is here. Don't worry if you don't follow all the steps we will go though this step by step in the next two weeks!

```
% Grid and operators
Grid.xmin = 0; Grid.xmax = Length; Grid.Nx = 30;
Grid.ymin = 0; Grid.ymax = 1;  Grid.Ny = 1;
Grid = build_grid(Grid);
[D,G,C,I,M] = build_ops(Grid);
L = -K*D*G; fs = qp/b*ones(Grid.Nx,1);

% Set Dirichlet BC's
BC.dof_dir = [Grid.dof_xmin; Grid.dof_xmax];
BC.dof_f_dir = [Grid.dof_f_xmin; Grid.dof_f_xmax];
BC.g = ha(Grid.xc(BC.dof_dir),qp);
[B,N,fn,BC] = build_bnd(BC,Grid,I);

% Solve linear boundary value problem
h = solve_lbvp(L,fs,B,BC.g,N);
q = comp_flux(D,K,G,h,fs,Grid,BC);
```
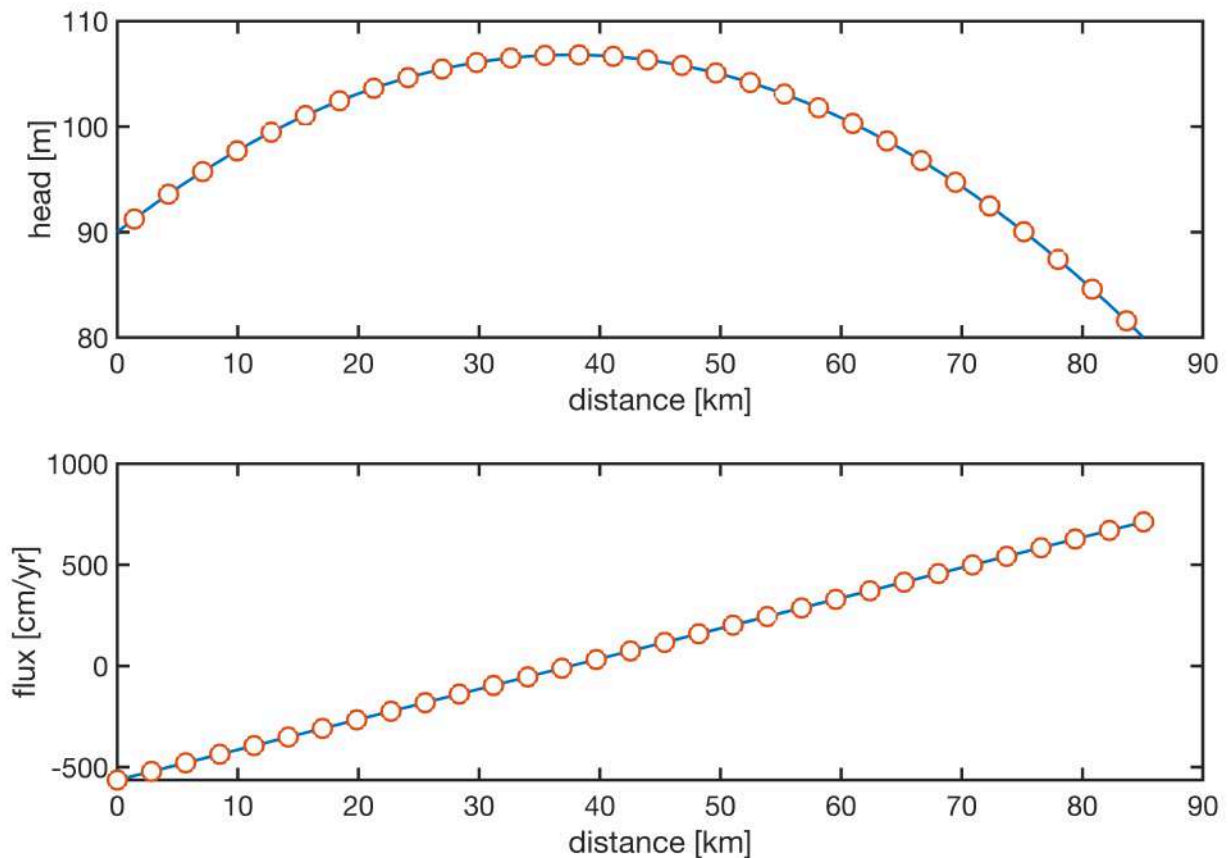
```matlab
figure('position',[10 10 900 600])
subplot 211
plot(xa/1e3,ha(xa,qp),'-'), hold on
plot(Grid.xc/1e3,h,'o','markerfacecolor','w')
xlabel 'distance [km]'
ylabel 'head [m]'

subplot 212
plot(xa/1e3,qa(xa,qp)/(cm2m/yr2s),'-'), hold on
plot(Grid.xf/1e3,q/(cm2m/yr2s),'o','markerfacecolor','w')
xlabel 'distance [km]'
ylabel 'flux [cm/yr]'
```


To solve the problem numerically we have to build the following addiional functions:

1. `build_bnd.m`: Builds matices and vectors assocated with the boundary conditons
2. `solve_lbvp.m`: Solves the linear system by imposing the boundary condtions
3. `comp_flux.m`: Computes the Darcy flux from the head values