Discrete operators in 1D

Best to discretize eqns in conservation form:

1) mass cons.: $\nabla \cdot \underset{\sim}{q} = f_s$

2) Darcy's law: $\underset{\sim}{q} = -K \nabla h$

Highlights two basic operators in vector calc.:

1) Divergence of a vector
2) Gradient of a scalar

Note: There is also the Curl but we won't need it.

Most PDE's in science and engineering are built from these operators!

If we had discrete analogs of these operators:

- solve different equations
- clean & readable implementation
- dimension & coordinat system independent

$\nabla \cdot, \nabla, (\nabla \times)$ are linear operators

$\Rightarrow$ matrices: $\underset{\approx}{D}, \underset{\approx}{G}, (\underset{\approx}{C})$

We are looking for two matrices so that

<u>continuous</u>: $\qquad\qquad$ <u>discrete</u>:

$$\nabla \cdot \underline{q} = f_s \quad \Rightarrow \quad \underline{\underline{D}} * q = \underline{f_s}$$

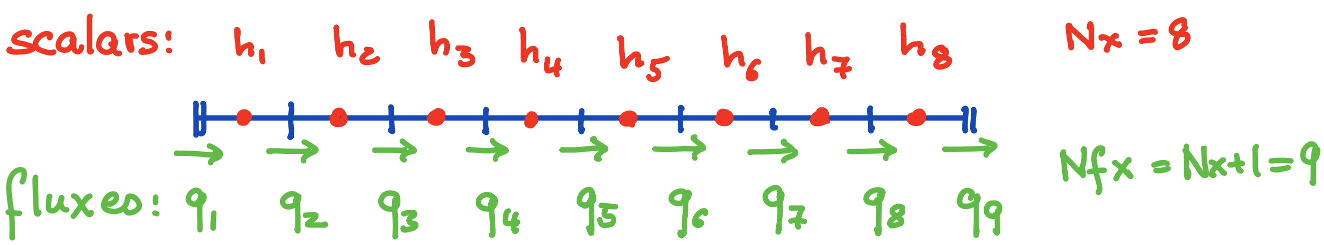
$$(k=1) \qquad \underline{q} = -\nabla h \quad \Rightarrow \quad q = -\underline{\underline{G}}\,\underline{h}$$

so that

$$-\nabla \cdot \nabla h = -\nabla^2 h = f_s \Rightarrow -\underline{\underline{D}}\,\underline{\underline{G}}\,\underline{h} = \underline{f_s}$$

Staggered grid

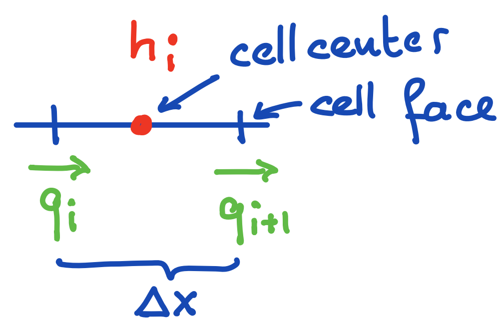
Needed to obtain a <u>compact</u> stencil.



Divide domain $x \in [0, L]$ into $N_x = 8$ control volumes of length Δx.

Control volume/cell:

i = degree of freedom (dof)

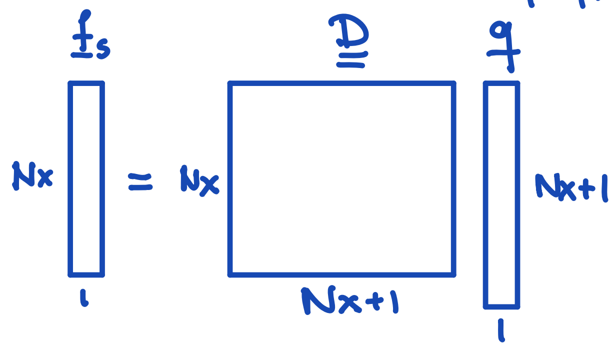


Discrete divergence operator

Divergence takes a flux and returns a scalar: $\nabla \cdot \underline{q} = f_s$

$$\underline{f_s} = \underline{\underline{D}} \ \underline{h}$$

$N_x \cdot 1 \quad (N_x+1) \cdot N_x \quad N_x \cdot 1$



$\Rightarrow$ $\underline{\underline{D}}$ is not square

maps from faces to centers

Entries into $\underline{\underline{D}}$ for $N_x = 8$

$$\underbrace{\begin{bmatrix} \\ \\ \\ f_4 \\ \\ \\ \\ \\ \end{bmatrix}}_{\underline{f_s}} = \frac{1}{\Delta x} \underbrace{\begin{bmatrix} -1 & 1 & & & & & & & \\ & -1 & 1 & & & & & & \\ & & -1 & 1 & & & & & \\ & & & -1 & 1 & & & & \\ & & & & -1 & 1 & & & \\ & & & & & -1 & 1 & & \\ & & & & & & -1 & 1 & \\ & & & & & & & -1 & 1 \end{bmatrix}}_{\underline{\underline{D}}} \underbrace{\begin{bmatrix} \\ \\ \\ q_4 \\ q_5 \\ \\ \\ \\ \\ \end{bmatrix}}_{\underline{q}}$$

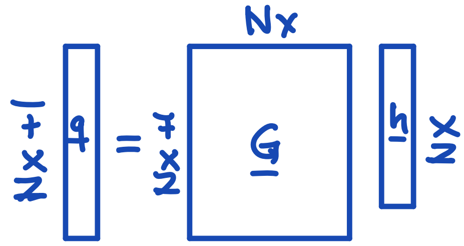
Finite Diff.: $\nabla \cdot q = \dfrac{dq}{dx} \approx \dfrac{q_{i+1} - q_i}{\Delta x} = f_i$

Example: $f_4 = \dfrac{q_5 - q_4}{\Delta x}$

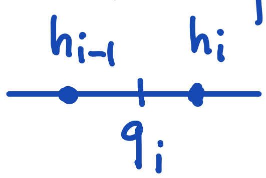
Discrete gradient operator

Gradient takes a scalar and returns a flux: $q = -\nabla h$

$$\underset{(N_x+1)\cdot 1}{\underline{q}} = - \underset{(N_x+1)\cdot N_x}{\underline{\underline{G}}} \; \underset{N_x \cdot 1}{\underline{h}}$$



Finite difference: $q_i = \dfrac{h_i - h_{i-1}}{\Delta x}$



Note: the minus sign is not part of $\underline{\underline{G}}$!

$$\underline{q} \quad \underline{\underline{G}} \quad \underline{h}$$

$$\begin{bmatrix} \\ \\ q_3 \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} = -\frac{1}{\Delta x} \begin{bmatrix} 0 & & & & & & & \\ -1 & 1 & & & & & & \\ & -1 & 1 & & & & & \\ & & -1 & 1 & & & & \\ & & & -1 & 1 & & & \\ & & & & -1 & 1 & & \\ & & & & & -1 & 1 & \\ & & & & & & -1 & 1 \\ & & & & & & & 0 \end{bmatrix} \begin{bmatrix} \\ h_2 \\ h_3 \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Example: $q_3 = -\dfrac{h_3 - h_2}{\Delta x}$

On boundaries we set flux to zero (natural BC)

Relation between $\underline{\underline{D}}$ and $\underline{\underline{G}}$

If we look at $\underline{\underline{D}}$ and $\underline{\underline{G}}$ we observe

$$\boxed{\underline{\underline{G}} = -\underline{\underline{D}}^T} \quad \text{in the interior of domain}$$

At bnd's the natural BC's in $\underline{\underline{G}}$ lead to difference.

This relation ship is due to the fact that $\nabla\cdot$ and ∇ are adjoint operators.

Discrete Laplacian Operator

Continuum: $\nabla\cdot\nabla = \nabla^2 \qquad -\nabla^2 h = f_s$

Discrete: $\underline{\underline{D}}\,\underline{\underline{G}} = \underline{\underline{L}} \qquad -\underline{\underline{L}}\,\underline{h} = \underline{f}_s$

Note: Laplacian takes scalar and returns a scalar

$\Rightarrow$ $\underline{\underline{L}}$ is Nx by Nx square matrix

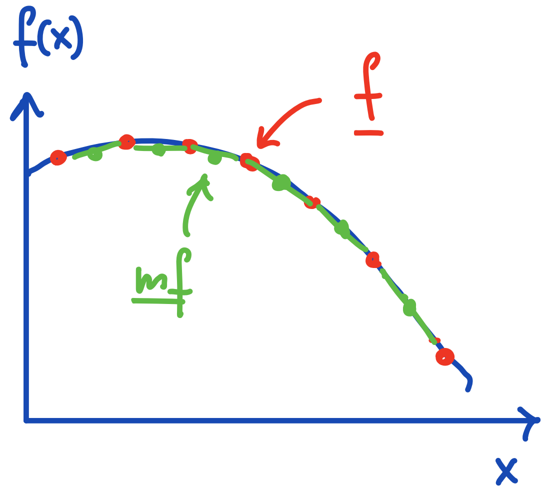
$$\underset{Nx\cdot Nx}{\underline{\underline{L}}} = \underset{Nx\cdot(Nx+1)}{\underline{\underline{D}}} \quad \underset{(Nx+1)\cdot Nx}{\underline{\underline{G}}}$$

Note: Rows of $\underline{\underline{D}}, \underline{\underline{G}}, \underline{\underline{L}}$ always sum to zero.

$$\underline{\underline{L}} = \frac{1}{\Delta x^2}\begin{bmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -1 \end{bmatrix}$$

Discrete mean operator

This will become useful once we have variable coefficients, $K = K(\underline{x})$.



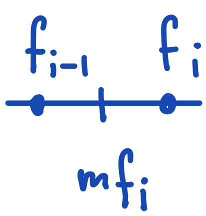
$\underline{\underline{M}}$ computes the arithmetic mean of cell center values on the faces.

$$\underset{(Nx+1)\cdot 1}{\underline{mf}} = \underset{(Nx+1)\cdot Nx}{\underline{\underline{M}}} \quad \underset{Nx \cdot 1}{\underline{f}}$$

$\Rightarrow$ $\underline{\underline{M}}$ has shape of $\underline{\underline{G}}$ (cellcenter $\rightarrow$ faces) but entries are different

$$\begin{bmatrix} \\ \\ mf_3 \\ \\ \\ \\ \\ \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & 1 & 1 & \\ & & & & 1 & 1 \\ & & & & & 2 \end{bmatrix} \begin{bmatrix} \\ f_2 \\ f_3 \\ \\ \\ \\ \end{bmatrix}$$

$\underline{mf}$ $\underline{f}$



for example: $mf_3 = \dfrac{f_2 + f_3}{2}$