Lecture 10: Radial coordinate systems

Logistics: - HW 4 due on Th

- Afzal office hrs? Mon 1:30 – 2:30pm
  Afzal's zoom

Last time: Fluxes and flux BC

- Sign convention $q_b > 0 \rightarrow$ inflow
- Neumann BC
  → convert flux into source term

$$f_n = q_b \frac{A}{V}$$

sign automatically correct

solve: $\boxed{\underline{\underline{L}}\,\underline{h} = \underline{f}_s + \underline{f}_n}$

need to specify both faces and cells (A & V)

dof_ncn   dof_f_ncn

Today: - Reconstruction of bnd fluxes

- Radial Coordinate systems

Flux computation

$$\boxed{\underline{q} = - \underline{\underline{Kd}} * \underline{\underline{G}} * \underline{h}}$$

works on interior
but not on bnd
because $\underline{\underline{G}}\,\underline{h} = \underline{0}$ on bnd

Residual: $\underline{r}(\underline{h}) = \underline{\underline{L}} * \underline{h} - \underline{f_s} = 0$

zero if linear system is solved correctly

$$\underline{\underline{L}}\,\underline{h} = \underline{f_s} + \underline{f_n}$$

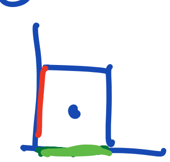
$\Rightarrow$ on Neu. bnd $\quad \underline{r} = \underline{f_n}$

residuals in bnd cells are source/sink term

in bnd cells: $\quad \underline{r} = q_b \dfrac{A}{V}$;

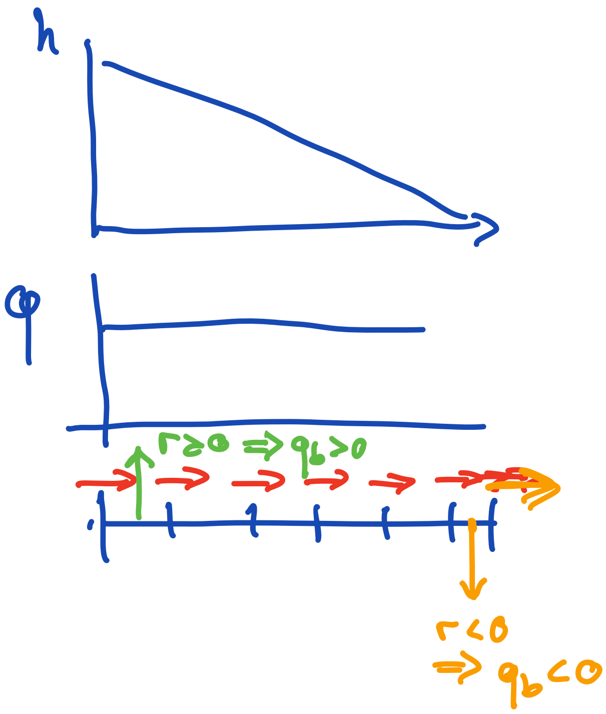
solve for $\boxed{q_b = r \dfrac{V}{A}}$ for single bnd cell

This is also true on dirichlet BC.
The only assumption is that each bnd cell has only one bnd face with a non zero flux

In 2D:  we'll set one face to no-flow

Sign change

We want q_b to have a sign that fits with rest of fluxes in the interior



here all fluxes are positive because they are in x-dir

$$q_b = r\frac{V}{A}$$

$f_u > 0$ inflow

- on xmin side sign fits
- on xmax side sign is reversed because out flow is negative residual

$\Rightarrow$ need to change sign on xmax bnd:

xmin: $q_b = r\frac{V}{A}$

xmax: $q_b = -r\frac{V}{A}$

Implementation

Function comp_flux_gen.m

define two anonymous functions:

$flux = @(h) \quad -\underline{\underline{Kd}} * \underline{\underline{G}} * \underline{h}; \quad \rightarrow flux(\underline{h})$

$res = @(h, cell) \quad \underline{\underline{L}}(cell,:) * \underline{h} - f_s(cell)$

vector of bnd cell

⇒ given to comp_flux_gen.m to allow us to change equations.

Inside ~~the~~ function we need two vectors:

dof_cell : column vector containing all bnd cells

dof_face: column vector containing all associated faces

Both vectors are same length, because only 1 bnd face is associated each bnd cell.

Reconstruct all bnd fluxes in one line:

Grid.V

$$\boxed{q(\text{dof\_face}) = \text{sign} \;.*\; \text{res}(h, \text{dof\_cell}) \;.*\; V(\text{dof\_cell}) \;./\; A(\text{dof\_face})}$$

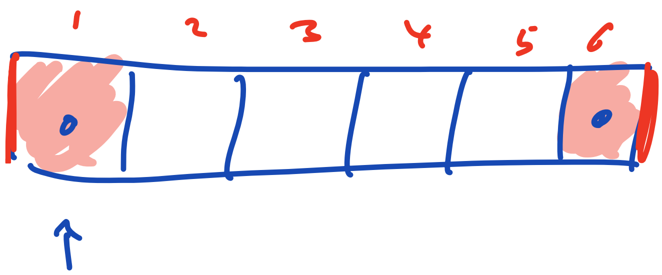
Grid.A

$$\text{sign} = \begin{cases} 1, & \text{dof\_face} \in \text{min bnd} \\ -1, & \text{dof\_face} \in \text{max bnd} \end{cases}$$

sign has to have same length as dof_cell

To check which side a dof is on use

ismember.m



dof_cell = [1; 6]   dof_xmin   dof_xmax

dof_face = [1, 7]   dof_fxmin ....

Radial coordinates

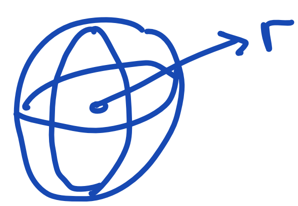
What is advantage of dyadic notation

$\nabla \cdot \quad \nabla \quad \nabla \times$?

Hides details of coordinate system and dimension

Even in one-dimension we have at least three possible coordinate systems:

1) cartesian linear
2) cylindrical radial
3) spherical radial



In general div, grad, curl change as you with coordinate system.

But for radial dir.: the gradient remains the same: $\nabla = \frac{d}{dx}$ $\quad x \to$ one coord dir (r)

In contrast divergence changes:

- linear: $\nabla\cdot = \frac{d}{dx}$ $\qquad d=1$
- cylindrical: $\nabla\cdot = \frac{1}{x}\frac{d}{dx}x$ $\qquad d=2$
- spherical: $\nabla\cdot = \frac{1}{x^2}\frac{d}{dx}x^2$ $\qquad d=3$

General radial divergence in d dimensions:

$$\boxed{\nabla\cdot = \frac{1}{x^{(d-1)}}\frac{d}{dx}x^{(d-1)}}$$

Geometric interpretation of radius terms.

GW flow: $\nabla\cdot q = f_s$

$$\frac{1}{x^{d-1}}\frac{d}{dx}\left(x^{d-1}q\right) = f_s$$

- the x^{d-1} on "inside" of divergence multiplies q $\Rightarrow$ represents growth of surface are as radius increase
  $\Rightarrow$ evaluate at face locations Grid.xf

$$\frac{d}{dx}\left(x^{d-1}q\right) = x^{d-1}f_s$$

- the $\frac{1}{x^{d-1}}$ on "outside" of divergence multiles all volumetric terms ($\underline{f}_s$) and accounts for the increase in cell volume with increasing radius
  $\Rightarrow$ evaluate at cell center Grid.xc

Discretization of radial divergence

$\underline{\underline{D}}$ = standard 1d linear divergence

general radial divergence:

$\underline{\underline{D}} = \underline{\underline{Rinv}} * \underline{\underline{D}} * \underline{\underline{R}}$

$\underline{\underline{R}}$ = spdiags(Grid.xf$^{(d-1)}$, 0, Nfx, Nfx)

$\underline{\underline{Rinv}}$ = spdiag(1./Grid.xc$^{(d-1)}$, 0, Nx, Nx)