# Lecture 13: Streamfunction

Logistics: - HW5 is due 10/11 !

- HW6 will be posted

Last time: - Discretization in 2D Kron

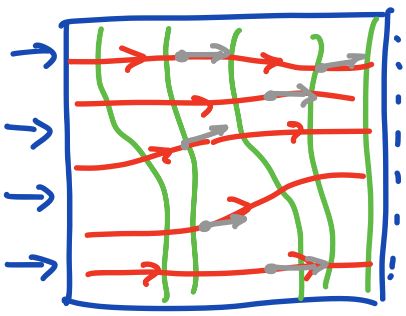
$$\underline{\underline{D}}_y = \underline{\underline{I}}_x \otimes \underline{\underline{D}}_y$$
$$\underline{\underline{D}}_x = \underline{\underline{D}}_x \otimes \underline{\underline{I}}_y$$
$$\underline{\underline{D}} = [\underline{\underline{D}}_x, \underline{\underline{D}}_y]$$

$$\underline{\underline{G}} = -\underline{\underline{D}}^T \quad \text{(interior)}$$

$$\underline{\underline{M}}_x = \underline{\underline{I}}_x \otimes \underline{\underline{M}}_x$$
$$\underline{\underline{M}}_y = \underline{\underline{M}}_y \otimes \underline{\underline{I}}_y$$
$$\underline{\underline{M}} = \begin{bmatrix} \underline{\underline{M}}_x \\ \underline{\underline{M}}_y \end{bmatrix}$$

update: build_grid build_ops

Today: Stream Lines & Stream function



Flow net

- head contours
- stream lines

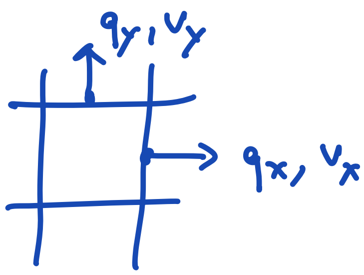
Streamlines: tangent to velocity field

## Streamline ODE system

$$\frac{dx}{dt} = v_x(\underline{x})$$

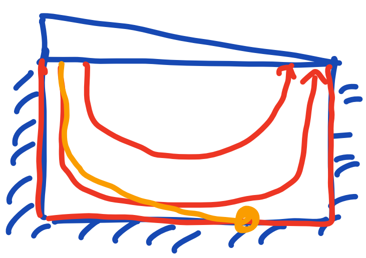
$$\frac{dy}{dt} = v_y(\underline{x})$$

$$\underline{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \qquad \underline{v} = \frac{\underline{q}}{\phi}$$

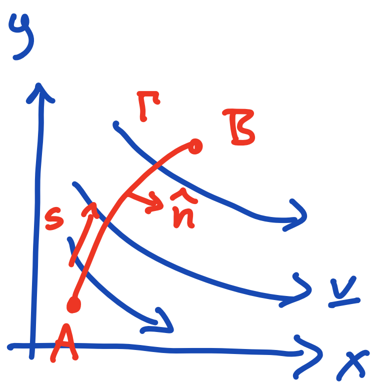



Matlab:

- streamline.m
  plot stream lines
  ⇒ average $q_x$ & $q_y$ to cell centers
- quiver → velocity arrows



## Different way of thinking about streamlines




compute cumulative flux between A & B

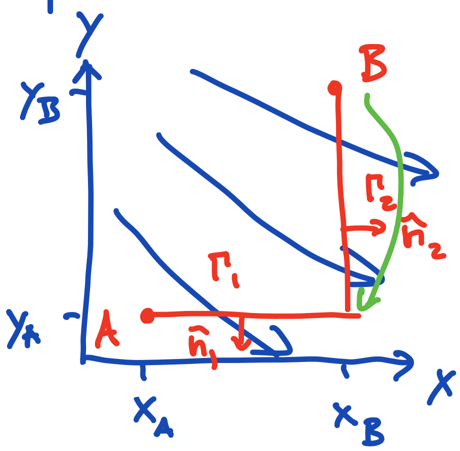
Γ is path

s is arclength variable

$\hat{n}$ is the right hand normal

$\boxed{\psi = \int_{\Gamma} \underline{v} \cdot \hat{n} \, ds}$ $\quad$ $\psi(\underline{x})$ is stream function

In the absence of any fluid sources or sinks between A & B, $\psi$ should not depend on path:




$\Rightarrow$ choose a path that simplifies integration.

along $\Gamma_1$: $\underline{v} \cdot \hat{n}_1 = -v_y$

$$(v_x \; v_y)\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

along $\Gamma_2$: $\underline{v} \cdot \underset{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\hat{n}_2} = v_x$

rewrite the integral:

$$\psi = \underbrace{\int_{x_A}^{x_B} -v_y(x, y_A)\, dx}_{\Gamma_1} + \underbrace{\int_{y_A}^{y_B} v_x(x_B, y)\, dy}_{\Gamma_2}$$

suppose $y_A = y_B$

$$\psi = \int_{x_A}^{x_B} -v_y \, dx \overset{\text{F.T.C}}{=} \int_{x_A}^{x_B} \frac{\partial \psi}{\partial x} dx \Rightarrow \frac{\partial \psi}{\partial x} = -v_y$$

suppose: $x_A = x_B$ :


$$\psi = \int_{y_A}^{y_B} v_x \, dy \overset{F.T.C}{=} \int_{y_A}^{y_B} \frac{\partial \psi}{\partial y} \, dy \Rightarrow \frac{\partial \psi}{\partial y} = v_x$$

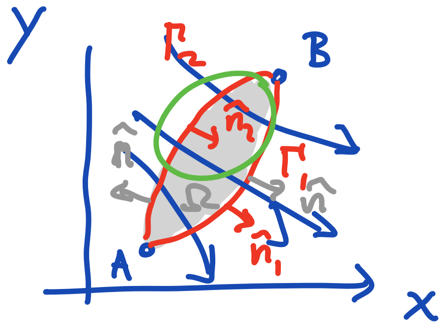
Therefore $\boxed{\frac{\partial \psi}{\partial x} = -v_y \qquad \frac{\partial \psi}{\partial y} = v_x}$

Often given as definition of stream function

Physical Interpretation:

- Change in cumulative flux in x-dir is proportional to the <u>negative</u> velocity in y-dir
- change in cumulative flux in y-dir is proportional to the velocity in x-dir.

These conclusions hold if the integral for $\psi$ is path independent.




$$\int_{\Gamma_1} \underline{v}\cdot\hat{n}_1\, ds = \int_{\Gamma_2} \underline{v}\cdot\hat{n}_2\, ds \quad ?$$

$$\int_{\Gamma_1} \underline{v}\cdot\hat{n}_1\, ds - \int_{\Gamma_2} \underline{v}\cdot\hat{n}_2\, ds = 0$$

Combine $\Gamma_1 + \Gamma_2 = \Gamma$ and define outside normal $\hat{n}$ to the enclosed area $\Omega$

$\hat{n} = \hat{n}_1$ on $\Gamma_1$ but $\hat{n} = -\hat{n}_2$ on $\Gamma_2$

$$\int_{\Gamma_1} \underline{v}\cdot\hat{n}\, ds + \int_{\Gamma_2} \underline{v}\cdot\hat{n}\, ds = \oint_{\Gamma} \underline{v}\cdot\hat{n}\, ds$$

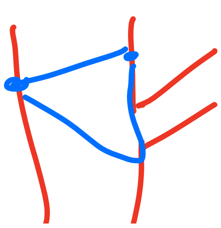
Apply the divergence theorem:

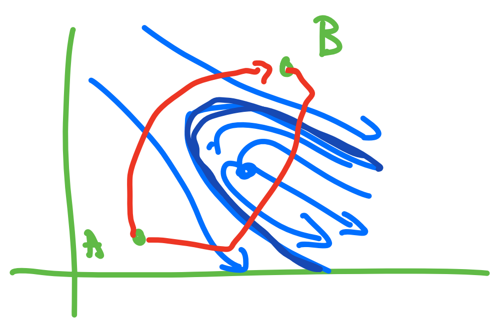
$$\oint_{\Gamma} \underline{v}\cdot\hat{n}\, ds = \int_{\Omega} \nabla\cdot\underline{v}\, dA = 0$$

Hence the stream function $\psi$ is well defined if $\boxed{\nabla\cdot\underline{v} = 0} \Rightarrow \boxed{\nabla\cdot\underline{q} = 0}$

$f_s$

- incompressible
- no sources or sinks

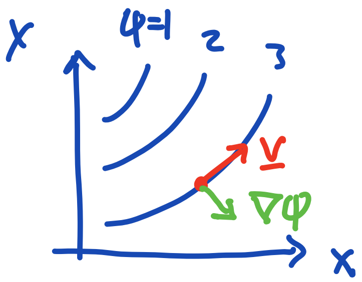




In an incompressible flow without sources/sinks the cummulative flux $\psi$ is a unique function of $\underline{x}$ and called the stream function.

What is the relation between the streamlines and the stream function?

1) The level sets (contours) of $\psi$ are tangential to the velocity vector and hence the streamlines.




$$\nabla\psi \cdot \underline{v} = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right) \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

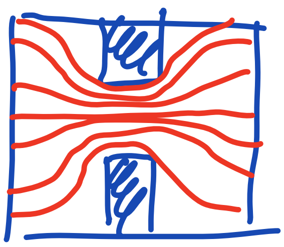
$$= (-v_y, v_x) \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix} =$$

$$= -v_y v_x + v_y v_x = 0 \quad \checkmark$$

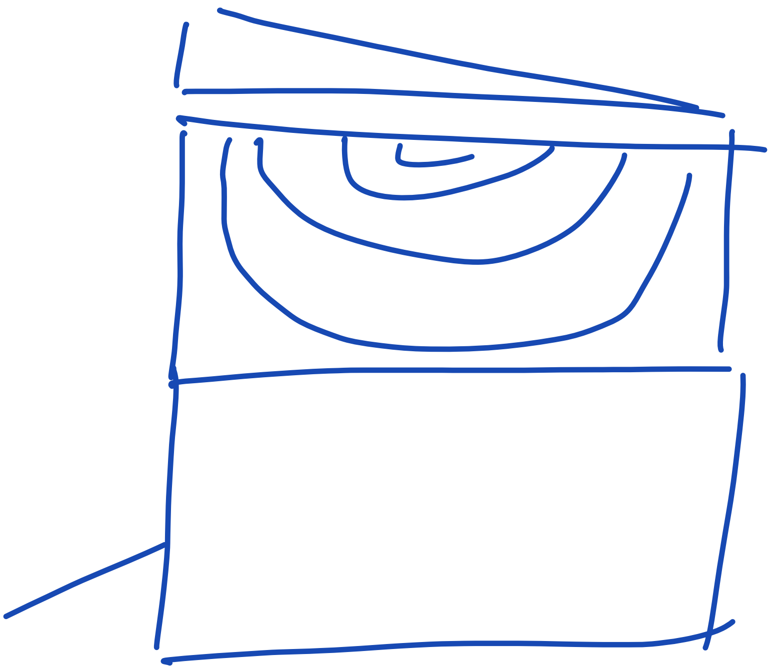
$$\Rightarrow \nabla\psi \perp \underline{v}$$

2) The magnitude of the velocity is equal to the magnitude of $\nabla\psi$.

$$|\nabla\psi| = \sqrt{(-v_y)^2 + v_x^2} = \sqrt{v_y^2 + v_x^2} = |\underline{v}|$$

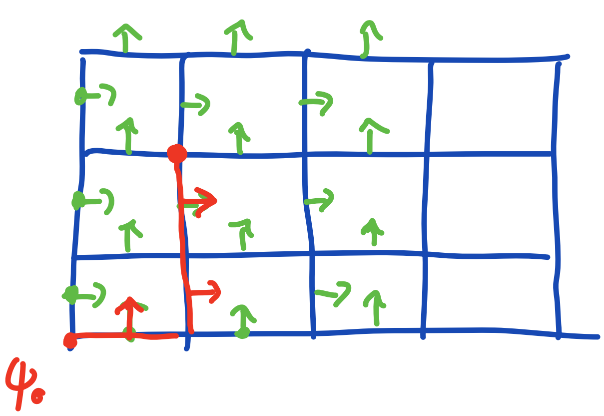


If we plot equally spaced contours of $\psi$ the spacing indicates the speed of the flow.



## Computing the streamfunction numerically

Definition: $\psi(x,y) = \psi_0(x_0,y_0) - \int_{x_0}^{x} v_y(x',y_0)\,dx' + \int_{y_0}^{y} v_x(x_0,y')\,dy'$




Given the location of $\varphi$ / $\underline{v}$ on the cell faces the natural location for $\psi$ is on the cell corners.

$\Rightarrow$ To compute $\psi$ we simply integrate $\phi$ along the faces of the cells.