Lecture 16: Steady heat equation

Logistics: HW7 due Th

Last time: - Energy balance

$$e = \phi \rho_f c_{p,f} T_f + (1-\phi)\rho_s c_{p,s} T_s$$

local thermal: $T_f = T_s \equiv T$

- Energy equation

$$\overline{\rho c_p} \frac{\partial T}{\partial t} + \nabla \cdot (\underbrace{q \rho_f c_{p,f} T}_{\text{advective}} - \underbrace{\bar{\kappa} \nabla T}_{\text{conductive}}) = \hat{f}_s$$

$\overline{\rho c_p} = \text{const.}$

$$\frac{\partial T}{\partial t} + \nabla \cdot [\underline{v}_e T - \bar{\alpha} \nabla T) = \frac{\hat{f}_s}{\overline{\rho c_p}}$$

$$\underline{v}_e = \frac{\phi \rho_f c_{pf}}{\overline{\rho c_p}} < 1$$

$$\bar{\alpha} = \frac{\bar{\kappa}}{\overline{\rho c_p}} \quad \text{thermal diffusivity}$$

Today: Steady heat conduction

$\Rightarrow$ Crustal Geotherm

Heat conduction

General energy ~~eqn~~ balance equ:

$$\overline{\rho c_p}\,\frac{\partial T}{\partial t} + \nabla \cdot \left[q \rho_f c_{pf} T - \bar{\kappa} \nabla T \right] = \hat{f}_s$$

$$\overline{\rho c_p} = \phi \rho_f c_{pf} + (1-\phi)\rho_s c_{p,s}$$

$$\bar{\kappa} = \phi \kappa_f + (1-\phi) \kappa_s$$

Consider the limit of pure rock:

$\phi = 0$, $\quad q = \underline{0}$ $\quad$ and $\quad \rho_s = \rho \quad c_{p,s} \equiv c_p \quad \kappa_s \equiv \kappa$

add radiogenic source term

$\hat{f}_s = \rho H$ $\qquad$ H = radiog. heat prod. $\left[\frac{W}{kg}\right]$

$\Rightarrow$ Heat equation

$$\boxed{\rho c_p \frac{\partial T}{\partial t} - \nabla \cdot [\kappa \nabla T] = \rho H}$$

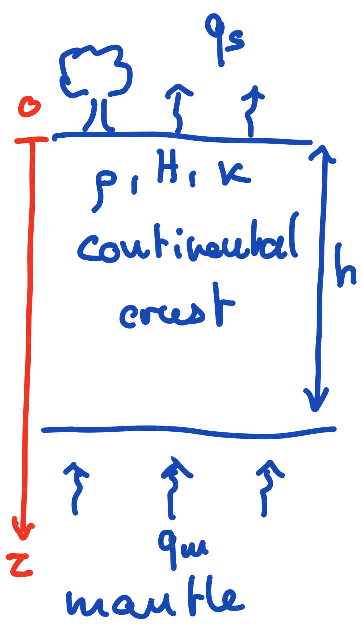
- Transient $\rightarrow$ changes with time
- radiogenic heat term

Steady heat conduction – Crustal Geotherm

steady → no change in time $\quad \frac{\partial T}{\partial t} = 0$

$\Rightarrow \quad \boxed{-\nabla \cdot [k \nabla T] = \rho H} \quad$ Poisson Eqn

⇒ same equation we solved for groundwater flow



h = ave $\overset{\text{continental}}{\downarrow}$ crustal thickness

q_m = mantle heat flow

q_s = surface heat flow

Fouriers law: $\quad q = -k \nabla T$

How does T vary with depth?

We know approximately:

$q_s = 65 \cdot 10^{-3} \ \frac{W}{m^2} \qquad k = 3.35 \ \frac{W}{mK}$

$\rho = 2700 \ kg/m^3 \qquad H = 9.6 \cdot 10^{-10} \ \frac{W}{kg}$ (surface)

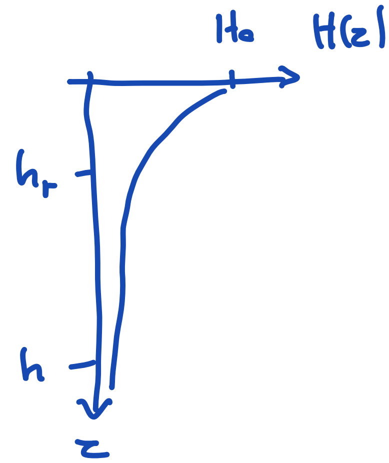
$h = 35 \ km$

We don't know q_m?

Heat production in crust

$$\int_0^h \rho H \, dz = \rho H h = 90.72 \cdot 10^{-3} \frac{W}{m^2} > q_s$$

Suggest H decays with depth:



$$\boxed{H(z) = \rho H_0 \exp(-z/h_r)}$$

H_0 = surface heat production

h_r = decay depth of heating

$h_r \ll h$

Here q_m and h_r are unknown

Relation between q_s and q_m

$$\nabla \cdot q = \rho H_0 \exp(-z/h_r)$$

$$\frac{dq}{dz} = \rho H_0 \exp(-z/h_r)$$

$$q_m - q_s = \rho H_0 \int_0^h \exp(-z/h_r) \, dz$$

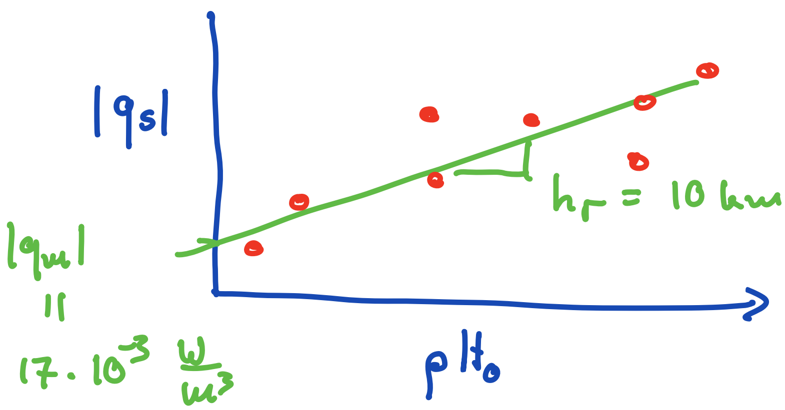
$$= -\rho H_0 h_r \left(\underbrace{\cancel{e^{-\frac{h}{h_r}}}}_{\ll 1} - \underbrace{\cancel{e^{0}}}_{1} \right) \qquad h_r \ll h$$

$$q_m - q_s \approx \rho H_0 h_r$$

$$q_m = -|q_m| \qquad q_s = -|q_s|$$

$$-|q_m| + |q_s| = \rho H_0 h_r$$

$$\Rightarrow \quad \underset{\substack{\uparrow \\ \text{measure}}}{|q_s|} \approx |q_m| + h_r \underbrace{\rho H_0}_{\substack{\uparrow \\ \text{measure}}}$$



$\Rightarrow$ all coefficients are determined, we can solve for geotherm $T(z)$

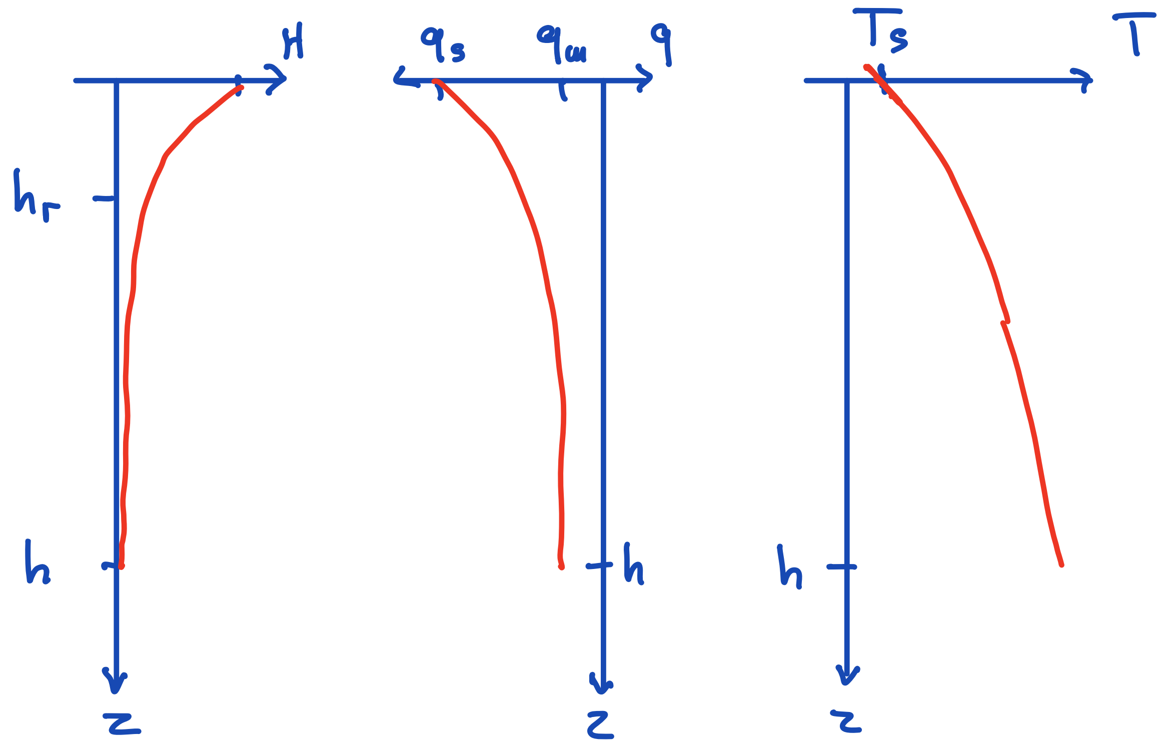
Geotherm heat conduction problem:

PDE: $\quad -k \dfrac{d^2T}{dz^2} = \rho H_0 e^{-\frac{z}{h_r}} \qquad z \in [0, h]$

BC: $\quad T(z=0) = T_s \qquad q \cdot n|_{z=h} = -|q_m|$

Integrate twice

$$\boxed{\begin{aligned} q(z) &= -|q_m| - \rho H_0 h_r \left(e^{-z/h_r} - e^{-h/h_r}\right) \\ T(z) &= T_s + \left(\frac{|q_m|}{k} - \frac{\rho H_0 h_r}{k} e^{-h/h_r}\right) + \frac{\rho H_0 h_r^2}{k}\left(1 - e^{-z/h_r}\right) \end{aligned}}$$



Numerical implementation

Continuous PDE: $-\nabla \cdot k \nabla T$

Discrete operator: $\underline{\underline{L}} = -\underline{\underline{D}}\ \underline{\underline{K_d}}\ \underline{\underline{G}}$

unknown $\underline{u} = T(\underline{x}_c)$

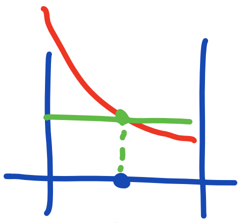
solve: $\underline{\underline{L}}\ \underline{u} = \underline{f}_s$

What about the source term?

Continuous : $f_s = \rho H_0 \, e^{-\frac{x}{h_r}}$ $\qquad x \to z$

Discrete : $f_s = f_s(x_c)$

This will converge with mesh refinement.



f_s should contain the averages of $f_s(x)$ over each cell

cell average: $\langle f_{s,i} \rangle = \frac{1}{\Delta x} \int_{x_{f,i}}^{x_{f,i+1}} \rho H_0 \, e^{-\frac{x}{h_r}} \, dx$

$$\boxed{\langle f_{s,i} \rangle = \frac{\rho H_0 h_r}{\Delta x} \left[\exp\left(-\frac{x_{f,i}}{h_r}\right) - \exp\left(-\frac{x_{f,i+1}}{h_r}\right) \right]}$$