Lecture 18: Heat Conduction

Logistics: - HW 8 will be posted on Th!

$\Rightarrow$ no HW this week

Last time: Heat equation

$$\boxed{\rho c_p \frac{\partial T}{\partial t} - \nabla \cdot [\kappa \nabla T] = \rho H} \qquad \underline{u} = T(\underline{x}_e)$$

Time discretization

$$\frac{\partial T}{\partial t} \approx \frac{\underline{u}^{n+1} - \underline{u}^n}{\Delta t} \qquad \Delta t = t^{n+1} - t^n$$

Theta method:

$$\underline{\underline{S}}(\underline{u}^{n+1} - \underline{u}^n) + \Delta t \, \underline{\underline{L}}(\theta \underline{u}^n + (1-\theta)\underline{u}^{n+1}) = \Delta t \, \underline{f}_s$$

Linear system: $\boxed{\underline{\underline{IM}}\, \underline{u}^{n+1} = \underline{\underline{EX}}\, \underline{u}^n + \Delta t \, \underline{f}_s}$

- Forward Euler: $\theta = 1$ explicit
- Backward Euler: $\theta = 0$ implicit
- Crank-Nicholson: $\theta = 0.5$ " $\Rightarrow 2^{nd}$

Amplification Matrix: $\boxed{\underline{u}^{n+1} = \underline{\underline{A}}^n \, \underline{u}^0}$

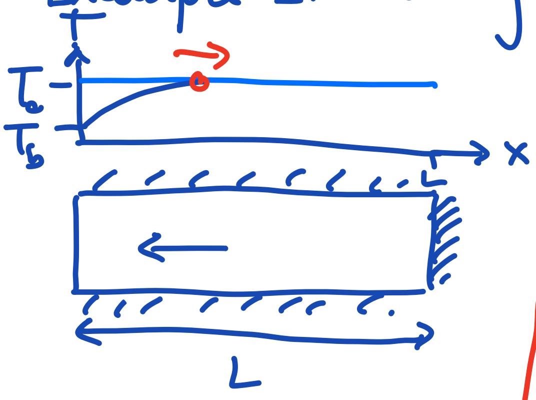
$$\underline{\underline{A}} = \boxed{\underline{\underline{IM}}^{-1} \underline{\underline{EX}}}$$

$\Rightarrow$ stability is determined by $\underline{\lambda}$'s of $\underline{\underline{A}}$!

Today: More on heat conduction!

Transient heat conduction

Example 1: Cooling of a finite bar



$D = \frac{\kappa}{\rho c_p}$

all prop. const: c_p, ρ, κ

$$\boxed{\begin{aligned} &\text{PDE}: \quad \frac{\partial T}{\partial t} - D \underbrace{\frac{\partial^2 T}{\partial x^2}}_{\nabla^2} = 0 \qquad x \in [0, L] \\ &\text{BC}: \quad T(x=0) = T_b \qquad \underline{q} \cdot \underline{n} = \frac{\partial T}{\partial x}\Big|_L = 0 \\ &\text{IC}: \quad T(x,0) = T_0 \end{aligned}}$$

Basic questions: 1) How does heat flow at bnd decay with time

2) How does change in T propagate into domain?

How many parameters: $\rho, c_p, \kappa, L, T_0, T_b$

$\Rightarrow$ 6 parameters

Non dimensionalize to reduce number of parameters?

$$T' = \frac{T - T_b}{T_0 - T_b} = \frac{T - T_b}{\Delta T} \qquad \Delta T = T_0 - T_b > 0 \qquad T_0 > T_b$$

$$x' = \frac{x}{L} \qquad t' = \frac{t}{t_c}$$

substitute: $\quad T = T_b + \Delta T \, T'$

$x = L x' \qquad t = t_c t'$

$$\frac{\cancel{\Delta T}}{t_c}\frac{\partial T'}{\partial t'} - \frac{D\cancel{\Delta T}}{L^2}\frac{\partial^2 T'}{\partial x'^2} = 0 \qquad x'\cancel{L} \in [0, \cancel{L}^{1}]$$

$$\frac{\partial T'}{\partial t'} - \underbrace{\frac{D t_c}{L^2}}_{\Pi = 1}\frac{\partial^2 T'}{\partial x'^2} = 0$$

$$\Pi = 1 \Rightarrow t_c = \frac{L^2}{D}$$

conductive time scale

$$= \frac{\partial T'}{\partial t'} - \frac{\partial^2 T'}{\partial x'^2} = 0 \qquad x' \in [0, 1]$$

BC: $T(x=0,t) = T_b$

$$\cancel{T_b} + \cancel{\Delta T}\, T'(x'=0, t') = \cancel{T_b} = 0$$

$$T'(0,t') = 0$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=L} = 0 \qquad \frac{\cancel{\Delta T}}{\cancel{L}}\left.\frac{\partial T'}{\partial x'}\right|_{x'\cancel{L}=\cancel{L}^{1}} = 0$$

$$\left.\frac{\partial T'}{\partial x'}\right|_{x'=1} = 0$$

IC: $T(x, t=0) = T_0$

$$T_b + \Delta T\, T'(x', t'=0) = T_0$$

$$\Delta T\, T'(x', 0) = T_0 - T_b = \Delta T$$

$$T'(x', 0) = 1$$

Dimensionless problem:

$$\boxed{\begin{array}{lll} \text{PDE:} & \dfrac{\partial T'}{\partial t'} - \dfrac{\partial^2 T'}{\partial x'^2} = 0 & x' \in [0,1] \\ \text{BC:} & T'(0,t') = 0 & \left.\dfrac{\partial T'}{\partial x'}\right|_{x'=1} = 0 \\ \text{IC:} & T'(x',0) = 1 & \end{array}}$$

no parameters left $\rightarrow$ <u>one</u> solution

<u>Analytic solution by separation of variables</u>

drop the primes: $T' \to T \quad x' \to x \quad t' \to t$

assume: $T(x,t) = h(x)\, g(t)$

substitute into PDE: $\dfrac{\partial T}{\partial t} = \dfrac{\partial^2 T}{\partial x^2}$

$$h(x) \frac{\partial g}{\partial t} = g(t) \frac{\partial^2 h}{\partial x^2}$$

separate variables

$$\frac{1}{g}\frac{\partial g}{\partial t} = \frac{1}{h}\frac{\partial^2 h}{\partial x^2} = -\lambda$$

Decomposes into two separate problems:

1) $\dfrac{dg}{dt} = -\lambda g$ IC: $T = g(t=0)\, h(x) = 1$

2) $\dfrac{d^2 h}{dx^2} = -\lambda h$ BC: $T = g\, h(0) = 0 \Rightarrow h(0) = 0$

$$\left.\frac{\partial T}{\partial x}\right|_1 = g \left.\frac{\partial h}{\partial x}\right|_1 = 0 \Rightarrow \left.\frac{\partial h}{\partial x}\right|_1 = 0$$

Time dependent problem: $\frac{dg}{dt} = -\lambda g$

$$\frac{dg}{g} = -\lambda\, dt$$

$$\ln(g) = -\lambda t + c$$

$$g = e^{-\lambda t + c} = a\, e^{-\lambda t} \qquad a = e^{+c}$$

if $\lambda > 0 \Rightarrow$ decay

const. a needs to be determined from IC.

Boundary value problem:

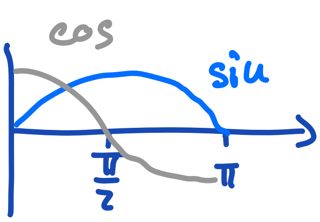
$$\frac{d^2h}{dx^2} = -\lambda h \qquad h(0) = 0 \qquad \left.\frac{dh}{dx}\right|_1 = 0$$

$$h(x) = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$$

$$\frac{dh}{dx} = \sqrt{\lambda}\left(c_1 \cos(\sqrt{\lambda} x) - c_2 \sin(\sqrt{\lambda} x)\right)$$

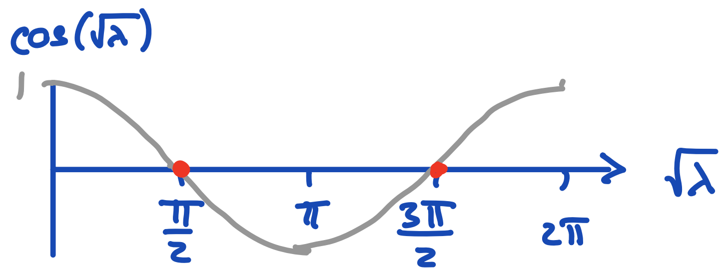
$$h(0) = c_1 \cancelto{c}{\sin(0)} + c_2 \cancelto{1}{\cos(0)} = 0$$

$$c_2 = 0$$



$$\Rightarrow \underline{h(x) = c_1 \sin(\sqrt{\lambda}\, x)}$$

$$\left.\frac{dh}{dx}\right|_1 = \sqrt{\lambda}\, c_1 \cos(\underline{\sqrt{\lambda}}) = 0 \qquad \text{multiple } \lambda\text{'s}$$



$\Rightarrow \sqrt{\lambda} \in [\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots]$ $\qquad \sqrt{\lambda} = (n-\frac{1}{2})\pi > 0$

$\boxed{\lambda_n = (n-\frac{1}{2})^2 \pi^2}$ $\qquad n \in [1, 2, 3 \dots]$

Solution is eigenfunction expansion:

$$T = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x)\, e^{-\lambda_n t}$$

$\Rightarrow$ need to determine A_n from initial cond.

$$T(x,0) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x)\, \cancel{e^{0}}^{\,1} = 1$$

Multiply by eigenfunction and integrate

$$\int_0^1 \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x) \sin(\sqrt{\lambda_m}\, x)\, dx = \int_0^1 \sin(\sqrt{\lambda_m}\, x)\, dx$$

orthogonality of sines:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

due to mixed BC we need:

$$\int_0^1 \sin\left((n-\tfrac{1}{2})\pi x\right) \sin\left((m-\tfrac{1}{2})\pi x\right) dx = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \end{cases}$$

Exchange sum & integral

$$\sum_{n=1}^{\infty} A_n \int_0^1 \sin(\sqrt{\lambda_n}\, x) \sin(\sqrt{\lambda_m}\, x)\, dx = \int_0^1 \sin(\sqrt{\lambda_m}\, x)\, dx$$

use orthogonality:

$$A_n \frac{1}{2} = \int_0^1 \sin(\sqrt{\lambda_n}\, x)\, dx$$

$$A_n = 2 \int_0^1 \sin\left((n-\tfrac{1}{2})\pi x\right) dx = \frac{4(\cancelto{0}{\sin(n\pi)}-1)}{\pi(1-2n)}$$

$$A_n = \frac{4}{(2n-1)\pi}$$

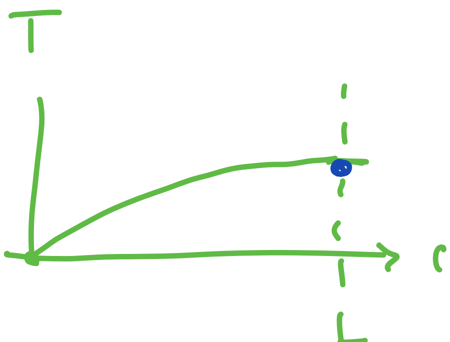
Dimensionless solution

$$\boxed{T'(x',t') = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left((n-\tfrac{1}{2})\pi x'\right) e^{-(n-\frac{1}{2})^2 \pi^2 t'}}$$

redimensionalize: $T' = \frac{T-T_b}{T_0-T_b}$ $\quad x' = \frac{x}{L}$ $\quad t' = \frac{tD}{L^2}$

$$\frac{T-T_b}{T_0-T_b} = \frac{4}{\pi}\Sigma$$

$$\boxed{T = T_b + (T_0-T_b)\,\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) e^{-\frac{(n-\frac{1}{2})^2\pi^2 D}{L^2}t}}$$



at late time only $n=1$ is left

$$\boxed{T \propto T_b + \Delta T\,\frac{2}{\pi}\sin\left(\frac{\pi x}{2L}\right) e^{-\frac{D\pi^2}{4L^2}t}}$$