Lecture 19: Heat conduction - infinite

Logistics: - HW8 will be posted today!
- No class next Tuesday / officehrs on Mon

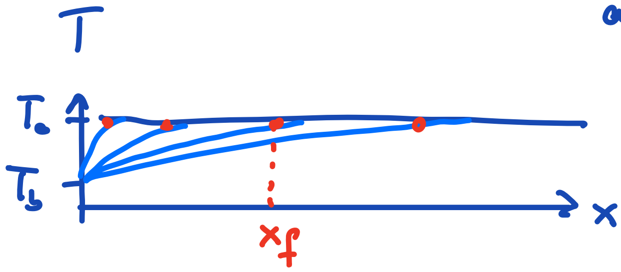
Last time: - Transient heat conduction
- Finite domain
- Non-dimensionalization 6 → 0
- Analytic solution by sep. of. variables

⇒ soln is eigenfunction expansion
- At late time → exponential decay

$\Sigma A_n \sin(n x)$

Today: • Infinite domain
⇒ Early time behavior
- Self-similar solution

Temperature propagation in semi-infinite slab

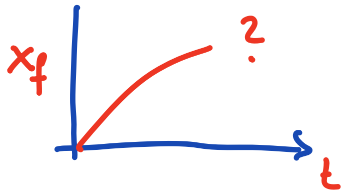


assume all prop. const.: ρ, φ, κ

PDE: $\dfrac{\partial T}{\partial t} = D \dfrac{\partial^2 T}{\partial x^2}$ $\quad x \geq 0$

BC: $T(0,t) = T_b$ $\quad \lim_{x \to \infty} T = T_0$

IC: $T(x,0) = T_0$



How fast does T front propagate?

Non-dimensionalize: $T' = \dfrac{T - T_b}{\Delta T}$ $\quad \Delta T = T_0 - T_b > 0$

$x' = \dfrac{x}{x_c}$ $\quad t' = \dfrac{t}{t_c}$

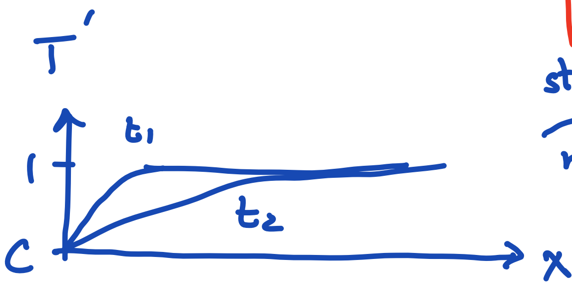
In finite domain: $x_c = L$

What scale to choose for x_c?

$D = \dfrac{\kappa}{\rho c_p}$ $\quad \left[\dfrac{L^2}{T}\right]$ $\Rightarrow$ $\sqrt{Dt}$ units of L !

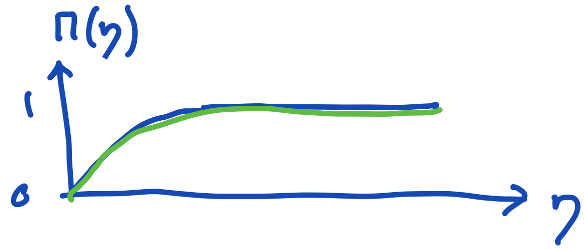
$x' \neq \dfrac{x}{\sqrt{Dt}}$ ~~two~~ this is function of x & t

$\Rightarrow$ new variable $\boxed{\eta = \dfrac{x}{\sqrt{4Dt}}}$ Boltzmann variable



streching

$\eta = \dfrac{x}{\sqrt{4Dt}}$



Solution is self-similar and η is the similarity variable. Reduce PDE $\rightarrow$ ODE

What is self-similar ODE?

Substitutions: $\eta = \dfrac{x}{\sqrt{4Dt}}$ $\qquad T' = \dfrac{T - T_b}{T_0 - T_b}$

into $\qquad \dfrac{\partial T}{\partial t} = D \dfrac{\partial^2 T}{\partial x^2}$

$$T'(x,t) = \Pi(\eta(x,t)) = \Pi(\eta)$$

transform derivatives:

$$\frac{\partial T'}{\partial t} = \frac{\partial}{\partial t}\Pi(\eta(x,t)) = \frac{d\Pi}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{\eta}{t}\frac{d\Pi}{d\eta}$$

$$\frac{\partial T'}{\partial x} = \frac{\partial}{\partial x}\Pi(\eta(x,t)) = \frac{d\Pi}{d\eta}\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4Dt}}\frac{d\Pi}{d\eta}$$

$$\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{4Dt}}\right) = \frac{1}{\sqrt{4Dt}}$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial}{\partial t}\left(\frac{x}{\sqrt{4D}}\, t^{-1/2}\right) = \frac{x}{\sqrt{4D}}\left(-\frac{1}{2}\,\underbrace{t^{-3/2}}_{\frac{1}{t\sqrt{t}}}\right) = -\frac{\overbrace{\frac{x}{\sqrt{4Dt}}}^{\eta}}{t} = -\frac{\eta}{2t}$$

substitute into PDE:

$$\frac{\partial T'}{\partial t} - D\frac{\partial^2 T'}{\partial x^2} = -\frac{\eta}{2t}\frac{d\Pi}{d\eta} - D\left(\frac{\partial \eta}{\partial x}\right)^2 \frac{d^2\Pi}{d\eta^2}$$

$$= -\frac{\eta}{2\cancel{t}}\frac{d\Pi}{d\eta} - \cancel{D}\,\frac{1}{4\cancel{D}\cancel{t}}\frac{d^2\Pi}{d\eta^2} = \underline{0}$$

Self-similar ODE:

ODE: $\dfrac{d^2\Pi}{d\eta^2} + 2\eta\dfrac{d\Pi}{d\eta} = 0$

BC: $\Pi(\eta=0) = 0 \qquad \lim\limits_{\eta\to\infty} \Pi(\eta) = 1$

Solve ODE:

1) substitution: $u = \dfrac{d\Pi}{d\eta} \;\rightarrow\; \dfrac{du}{d\eta} + 2\eta u = 0$

2) sep. of var: $\dfrac{du}{u} = -2\eta\, d\eta$

$$\ln u = -\eta^2 + a \;\rightarrow\; u = c\,e^{-\eta^2}$$

3) resubstitute: $\dfrac{d\Pi}{d\eta} = c\,e^{-\eta^2}$

4) sep. of var.: $d\Pi = c\,e^{-\eta^2}\,d\eta$

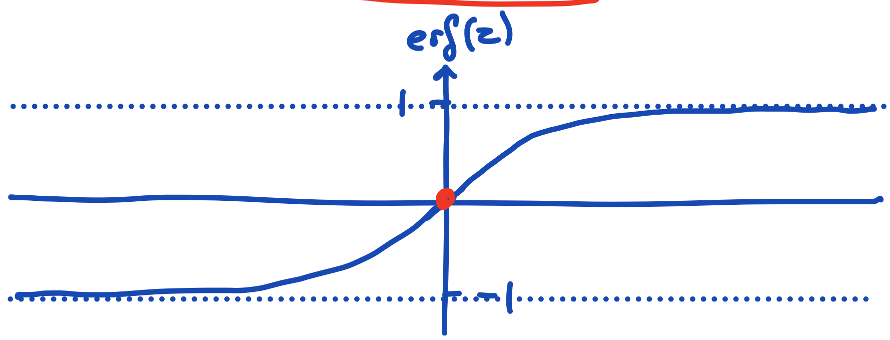
$$\int_{\Pi=0}^{\Pi(\eta)} d\Pi = c \int_{\eta=0}^{\eta} e^{-\eta'^2} \, d\eta'$$

$$\Pi(\eta) = c \int_0^{\eta} e^{-\eta'^2} \, d\eta'$$

cannot integrate analytically

5) Identify the error function

$$\boxed{\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^{z} e^{-z'^2} \, dz'}$$



properties: • $\lim_{z \to \infty} \mathrm{erf}(z) = 1$

• $\mathrm{erf}(-z) = -\mathrm{erf}(z)$ "point sym."

• $\mathrm{erf}(0) = 0$

• $\mathrm{erf}(z) \approx z \qquad |z| \ll 1$

Therefore $\quad \Pi(\eta) = c \frac{\sqrt{\pi}}{2} \mathrm{erf}(\eta)$

BC: $\lim_{y \to \infty} \Pi(y) = c \frac{\sqrt{\pi}}{2} 1 = 1 \Rightarrow c = \frac{2}{\sqrt{\pi}}$

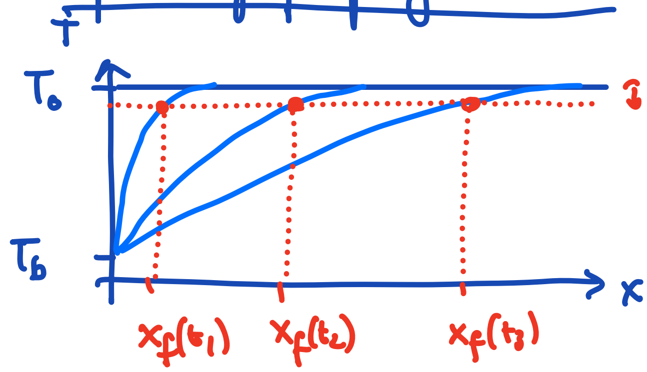
Self similar solution: $\boxed{\Pi(y) = \mathrm{erf}(y)}$

6) Resubstitute: $T' = \Pi(y) = \frac{T - T_b}{\Delta T}$ $\quad y = \frac{x}{\sqrt{4Dt}}$

Dimensional soln: $\boxed{T(x,t) = T_b + \Delta T \, \mathrm{erf}\left(\frac{x}{\sqrt{4Dt}}\right)}$

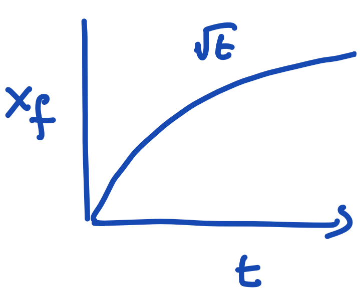
$\Delta T = T_0 - T_b$

Speed of propagation



looking for

$T(x_f, t) = T_0 - \varepsilon (T_0 - T_b)$



$$T_0 - \varepsilon \Delta T = T_b + \Delta T \, \mathrm{erf}\left(\frac{x_f}{\sqrt{4Dt}}\right)$$

$$\cancel{\Delta T}(1 - \varepsilon) = \cancel{\Delta T} \, \mathrm{erf}\left(\frac{x_f}{\sqrt{4Dt}}\right)$$

$$\frac{x_f}{\sqrt{4Dt}} = \underbrace{\mathrm{erf}^{-1}(1 - \varepsilon)}_{\alpha(\varepsilon) = \text{constant}}$$

$$\Rightarrow \qquad x_f = \alpha(\varepsilon)\sqrt{4Dt} \sim \sqrt{t}$$

for $\varepsilon = 0.1 \qquad \alpha \approx 1.16$