Lecture 2: Balance Laws

Logistics: Office hrs: Mon 2-3 pm

Wed 3-4 pm

Hydrothermal convection

Last time: - Porous media basics

volume fraction: $\phi_p = \frac{V_p}{V_T}$ $\quad V_T = \sum_p V_p$

porosity

Darcy's law: $\boxed{Q = -K A \frac{\Delta h}{\Delta \ell}}$

Rates vs. Fluxes: Q vs. q

scalar $\frac{\#}{T}$ $\quad$ vector $\frac{\#}{L^2 T}$ $\quad$ discharge $\frac{L^3}{T}$ $\quad$ spec. disch $\frac{L^3}{L^2 T} = \frac{L}{T}$

Today: - Darcy's law in flux form

- General balance law
- Fluid mass balance in porous media
- Incompressible flow in porous media

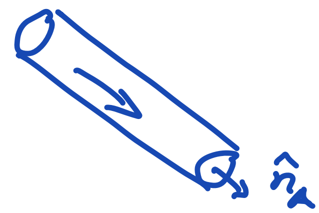
$\Rightarrow$ first equation to solve

Vector form of Darcy's law

From Experiments: $Q = -KA\frac{\Delta h}{\Delta L}$

$\uparrow$
Rate = Discharge

spec. discharge: $q = \frac{Q}{A}\hat{n}_A$

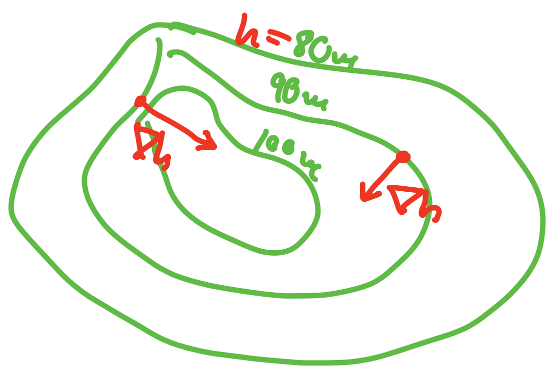


$$|q| = \frac{Q}{A} = -K\underbrace{\frac{\Delta h}{\Delta L}}_{\text{derivative } \Delta L \to 0}$$

$$|q| = -K\frac{dh}{dx} \qquad 1D$$

multi dimensional version: $\boxed{q = -K\nabla h}$

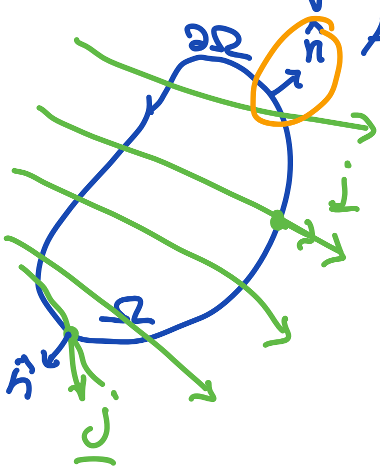
here $\nabla h = \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial z} \end{pmatrix}$



Balance laws

- most fundamental physical equs in sciences are balance laws!
- Balance law: accounts for change of some quantity

Derivation of general balance law



Account for change of unknown $u(x,t)$ in domain Ω due to either fluxes j accross boundary $\partial\Omega$ or production/consumption of u by source/sink term $\hat{f}$ in Ω.

Units of basic quantities:

- u is a density $\left[\frac{\#}{L^3}\right]$
- j is a flux $\left[\frac{\#}{L^2 T}\right]$
- $\hat{f}$ is a vol. source $\left[\frac{\#}{L^3 T}\right]$

General balance on Ω:

$$\boxed{\frac{dU}{dt} = J + F}$$

macroscopic balance

1) U is amount of u in Ω:

$$U(t) = \int_\Omega u(\underline{x},t)\, dV \qquad [\#]$$

2) J is the rate of transport of u across $\partial\Omega$ by $\underline{j}$:

$$\underset{[\frac{\#}{T}]}{J(t)} = -\int_{\partial\Omega} \underline{j}(\underline{x},t)\cdot\hat{n}\, \cancel{dt}\ dA$$

3) F is rate of prod./consumption of u inside Ω:

$$\underset{\frac{\#}{T}}{F(t)} = \int_\Omega \hat{f}(\underline{x},t)\, dV$$

substitute into balance:

$$\boxed{\frac{d}{dt}\int_\Omega u\, dV = -\int_{\partial\Omega} \underline{j}\cdot\hat{n}\, dA + \int_\Omega \hat{f}\, dV}$$

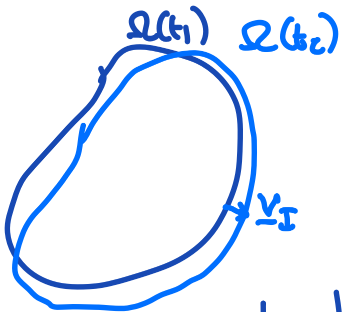
Integral balance law

$$\underbrace{\frac{1}{T}\ \frac{\#}{L^3}\ L^3}_{\frac{\#}{T}} \qquad \underbrace{\frac{\#}{L^2T}\ L^2}_{\frac{\#}{T}} \qquad \underbrace{\frac{\#}{L^3T}\ L^3}_{\frac{\#}{T}} \qquad \checkmark$$

To obtain local PDE we need:

1) Exchange the derivative and integral
2) Transform surface to a volume integral

1) Reynolds Transport Theorem



$$\boxed{\frac{d}{dt}\int_{\Omega(t)} u(\mathbf{x},t)\,dV = \int_{\Omega} \frac{\partial u}{\partial t}\,dV + \oint_{\partial\Omega} u\,(\mathbf{v}_I\cdot\hat{n})\,dS}$$

In this class we consider Eulerian limit where domain is fixed $\mathbf{v}_I = 0$

$$\Rightarrow \quad \frac{d}{dt}\int_{\Omega} u\,dV = \int_{\Omega} \frac{\partial u}{\partial t}\,dV$$

2) Divergence theorem: $\boxed{\int_{\partial\Omega} \mathbf{j}\cdot\hat{n}\,dA = \int_{\Omega} \nabla\cdot\mathbf{j}\,dV}$

Substitute into integral balance law:

$$\int_{\Omega}\underbrace{\left(\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{j} - \hat{f}\right)}_{g(\mathbf{x},t)}dV = 0$$

Localization: because Ω is arbitrary the only way for the integral to be always zero is that the integrand itself is zero.

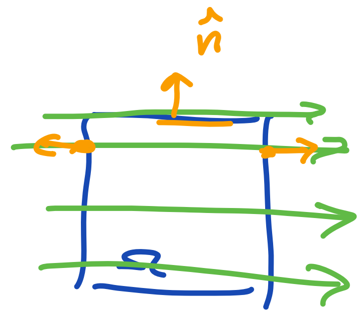
$$\boxed{\frac{\partial u}{\partial t} + \nabla \cdot \underset{\sim}{j} = \hat{f}}$$

Local form of general balance law

u = unknown $\left[\frac{\#}{L^3}\right]$

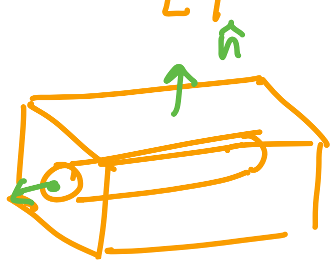
$\underset{\sim}{j}$ = flux $\left[\frac{\#}{L^2 T}\right]$

$\hat{f}$ = vol. rate $\left[\frac{\#}{L^3 T}\right]$



Difference between flux & velocity in porous med.

$\underset{\sim}{q} = -K \nabla h \qquad \frac{L^3}{L^2 T}$



$\underset{\sim}{v} = \frac{q}{\phi}$

Fluid Mass Balance:

Gen. balance law: $\frac{\partial u}{\partial t} + \nabla \cdot \underline{j} = \hat{f}$

1) Define unknown to be balanced

$u = \phi \rho$ $\qquad$ ρ = pore fluid density $\left[\frac{M}{L^3}\right]$

$\qquad\qquad$ ϕ = porosity $\left[\frac{L^3}{L^3} = 1\right]$

u is fluid mass per unit volume of porous medium

Note: assume porous medium is saturated

2) Define the mass flux of pore fluid

$\underline{j} = \rho \underline{q} = \underbrace{\rho \phi}_{u} \underline{v} = u \underline{v}$ $\qquad$ $\underline{q}$ = vol. flux $\frac{L^3}{L^2 T} = \frac{L}{T}$

$\qquad\qquad$ $\underline{v}$ = ave. intrst. velocity $\frac{L}{T}$

$\Rightarrow$ advective flux

fluid mass carried/advected through the porous medium by groundwater flow

3) Vol. source: $\hat{f} = \rho f$

Substitute

$$\boxed{\frac{\partial}{\partial t}(\phi\rho) + \nabla\cdot(\cancel{\rho} q) = \cancel{\rho} f}$$

fluid mass balance in porous medium

Constitutive laws:

1) Darcy's law: $q = -K\nabla h$

2) Equation of state: $\rho = \rho(h)$ $\qquad h = \frac{p}{\rho g}$

$\Rightarrow$ one equation for one unknown h

Incompressible flow:

$\rho = \text{const}$

$\phi = \phi(\underline{x})$ does not change with time

$\frac{\partial}{\partial t}(\rho\phi) = 0$

mass balance: $\nabla\cdot q = f$

Darcy's law: $q = -K\nabla h$

$$\Rightarrow \quad \boxed{-\nabla\cdot[K\nabla h] = f}$$

Poisson Equ